Fock Space, Factorisation and Beam Splittings: Characterisation and Applications in the Natural Sciences

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To:

The Light

The Truth

The Life

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M. Gäbler

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Introduction

In the second half of the 1920s, theoretical physicists like Jordan ([49]) and Dirac ([9]) developed the concept of so-called second quantisation: the passage from one-particle quantum systems to systems with an arbitrary number of particles. Building on these ideas, in his 1932 paper "Konfigurationsraum und zweite Quantelung" ([34]), the Russian physicist Vladimir Aleksandrovich Fock implicitely introduced two particular Hilbert spaces endowed with the structure to describe such many-particle systems, later specified and known as bosonic (symmetric) and fermionic (anti-symmetric) Fock spaces.

This work deals with configurations (ensembles) of bosons, i.e. symmetric quantum particles. This means that two quantum configurations are considered equal if they coincide except for their ordering: interchanging two or more particles of a fixed configuration cannot be distinguished.

A very important example for bosonic particles are photons. Symmetric Fock space therefore plays a vital role in quantum optics. This is in particular true since the developement of laser light in the 1960s. A beam of light produced by a laser is very close to the ideal of being coherent, meaning that there is a random number of non-interacting (independent) photons all being in the same one-particle state (phase, frequency and polarisation in a quantum sense). Such a coherent state can most conveniently be described by a wave function on the symmetric Fock space given by a normalised exponential vector.

Because all photons are independent, the marginal states of two "disjoint" parts of a coherent beam are also independent. In fact, as shown by Glauber/Titulaer in 1966 ([65]) for the class of all normal states and generalised to locally normal states by Fichtner/Schreiter in 1990 ([32]), coherent states are characterised by this property of so-called local independence. The classical analogon of a coherent state is a spatial Poisson process, which is also characterised by (classical) local independence (see [56]).

In order to describe local independence of states, symmetric Fock space needs to be "factorisable by regions" in the sense of tensor product of Hilbert spaces. Representing the bosonic Fock space as an L^2-space over symmetric point configurations as introduced by Fichtner/Freudenberg in 1982 (see [16], also [17, 18, 35, 19, 20]), which could be seen as a "quantisation of point processes", this property of regional factorisation is very natural: the underlying measure space of configurations is the product of the measure spaces corresponding to a decomposition into disjoint regions. For this representation neither finiteness nor non-atomicity of the underlying one-particle measure space is necessary (see [8, 51, 63]), which is quite an advantage in comparison with similar approaches by Guichardet (1972, see [44]) or Maassen (1984, see [55]).

If non-atomicity is assumed, it was shown by Araki/Woods (Araki-Woods embedding theorem 1966, see [3]) that, making some reasonable assumptions like commutativity and associativity, an algebraic version of this factorisation property is characteristic for symmetric Fock space: a Hilbert space is factorisable if and only if it can be isomorphically embedded into the symmetric Fock space over a one-particle Hilbert space that also decomposes, but in the sense of orthogonal sums. Factorisable vectors, in this algebraic setting for finite and non-atomic one-particle measure spaces, are also shown in Guichardet (1972, see [44]) to be multiples of exponential vectors and factorisable unitaries U on Fock space are of the kind

$$U = c \cdot \mathcal{W}(h)\,\Gamma(T)$$

for a complex number c of modulus 1, the Weyl operator corresponding to h and an operator of second quantisation of unitary T on the underlying one-particle space. In addition, T "acts locally" in the sense that it preserves all regional subspaces.

Now what kind of operators T have this property of "acting locally"?

Specialising to a one-particle Hilbert space of square-integrable, vector-valued maps on the one hand and also relaxing some requirements made in [44] on the other, this thesis, as its main achievement, provides a complete answer to this question. Using the approach of quantisation of point processes mentioned above, it is shown that, for a locally finite and non-atomic one-particle measure space, factorisable isometries U are still of the above type and the operators T are exactly the operators of matrix multiplication. If U is moreover vacuum-preserving, it becomes a general kind of beam splitting with an arbitrary number of beams of in- and output (there can be more beams of out- than input), thus emphasising the role of beam splittings as fundamental objects (similar to coherent states) of bosonic quantum field theory, quantum optics in particular.

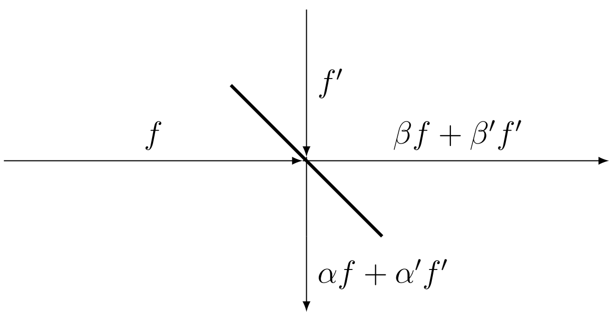

Figure 0.1: Beam Splitting with Two Beams of In- and Output

A beam splitting with two beams of both in- and output may be realised by a two-way-mirror. For suitable splitting rates $\alpha, \alpha', \beta, \beta'$, figure 0.1 illustrates how two beams of input consisting of photons in state f and f', respectively, are split (partially reflected and transmitted) into $\alpha f, \beta f$ and $\alpha' f', \beta' f'$ and then combined to yield beams of output where the photons are in states $\alpha f + \alpha' f'$ and $\beta f + \beta' f'$.

Besides the obvious application of beam splittings in quantum optics, they, at least as a mathematical object, have turned out useful in many other fields, for example:

1. quantum Markov chains in the sense of Accardi ([2, 1]): see Fichtner/ Freudenberg/Liebscher/Schubert 1994-2005 ([36, 22, 23, 24, 53, 63]),
2. exchange operators: Fichtner/Freudenberg/Liebscher 2004 ([25]),
3. quantum teleportation: Fichtner/Freudenberg/Ohya 2003-2005 ([26, 27, 59]),
4. quantum logical/communication gates: Freudenberg/Ohya/Turchina/ Watanabe 2000-2006 ([39, 38, 40, 41, 42, 37]),
5. brain models: Fichtner/Fichtner/Freudenberg/Gäbler/Ohya 2005-2010 ([29, 11, 12, 13, 14, 10, 33, 43]).

We will now sketch the content of this thesis.

Chapter 1 introduces the basic spaces, functions, operators and isomorphic representations needed for the description of factorisability in general and beam splittings in particular in chapters 2 and 3. Following the outline in [13] we review some basic concepts from quantum mechanics in section 1.1 (see also [47, 48]) and the theory of point processes in section 1.2 (see [62, 7]). Combining these ideas in section 1.3, i.e. quantising point processes according to [16, 17, 18], leads to the particular definition of the symmetric Fock space used in this book. Because working with exponential vectors lies at the core of this thesis, a detailed exposition on their properties will be given in section 1.4 including proofs, even though most of them are well-established (see [60, 58, 51]). Sections 1.5 and 1.6 conclude this introductory chapter with a representation of multiple quantum configurations on so-called multiple Fock space and how both single and multiple Fock space share the beautiful property of regional decomposition or regional factorisation.

Having seen factorisability of (multiple) Fock space in sections 1.5 and 1.6, chapter 2 is dedicated to the characterisation of factorisable elements of and isometric operators on multiple Fock space. The main results are summarised in section 2.1 and developed in detail in sections 2.2 and 2.3. In an algebraic setting, they can also be found in [3, 44]. Necessary auxiliary results are given in sections 2.4, 2.5 and 2.6. In particular, it is shown in section 2.4 that for non-atomic primary measures the so-called $\sum\!\!\!\!\!\!\int$-Lemma (see [54, 51, 63]) implies the existence of a covering of multiple Fock space by multiple configurations such that their superposition is both simple and finite (see corollary 2.4.4). This result was also used but not proven in [32]. Similar to the approximation by so-called toy exponentials or toy Fock space (see [57, 60]), we will approximate exponential vectors of the kind $\exp[zf + zg]$ in section 2.5, lemma 2.5.3. Section 2.6 recovers the characterisation of isometries that map exponential vectors to multiples of exponential vectors as found in [52] but using the more elementary methods from a specialised result in [44]. Using the approximation of exponential vectors from section 2.5 we also add a similar characterisation of (only) bounded operators in our particular setting of multiple Fock space (see proposition 2.6.6).

In chapter 3 we present the main result: definition and characterisation of beam splittings with an arbitrary number of beams of in- and output. After a brief excursion to the theory of operators of matrix multiplication (see [6, 46, 45]), i.e. a generalisation of operators of multiplication to spaces of vector-valued functions, we will also add to this theory by showing, that bounded operators are operators of matrix multiplication if and only if they preserve regional subspaces (see theorem 3.2.7). We will then be prepared to return to the original aim of defining and characterising beam splittings in definition 3.3.1 and theorem 3.3.5 of section 3.3. How beam splittings relate to so-called exchange operators (see [25]) will be shown in section 3.4. As studying factorisable operators, hence beam splittings, on multliple Fock space was motivated by a quantum model of recognition (see [29, 11, 12, 13, 14, 10, 33, 43, 15]), the role of a particular kind of beam splitting in this model is recapitulated in the concluding section 3.5 of this dissertation, thus making it part of the "century of life sciences" proposed by Ohya.

1 The Bosonic Fock Space

Symbolise as $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ the sets of natural, non-negative integer, integer, rational, real and complex numbers, respectively. The real part, imaginary part and complex-conjugate of $z \in \mathbb{C}$ will be denoted with $\operatorname{Re} z$, $\operatorname{Im} z$ and $\overline{z}$.

If $(G, \mathfrak{G})$ and $(G', \mathfrak{G}')$ are two measurable spaces, $\mathbb{M}(G, G')$ will denote the set of all measurable maps from G to G'. In case $G' = \mathbb{C}$ and $\mathfrak{G}'$ the Borel sets we will simply write $\mathbb{M}(G)$. $A^c = G \backslash A$ is the complement and χ_A the indicator function of $A \subset G$. The spaces of μ-equivalence classes of essentially bounded and square-integrable complex-valued measurable maps on $(G, \mathfrak{G}, \mu)$ will be denoted with $L^\infty(G)$ and $L^2(G)$, respectively.

Speaking of a Hilbert space H, we shall always mean a separable, complex Hilbert space with scalar product $\langle \cdot, \cdot \rangle_H$, which is linear in its second argument, and respective norm $\|\cdot\|_H$. The subscripts will be dropped, in case of no ambiguity about the Hilbert space in question. The symbol $\cong$ denotes isomorphic equivalence of Hilbert spaces, its elements or operators on them. The set of bounded operators on H is denoted $\mathcal{B}(H)$, the orthogonal projections onto a subspace $H' \subset H$ being $\mathrm{Proj}_{H'}$. In particular, we will use $\mathbb{1}_H$ for the identity operator and Proj_f in case $H' = \mathrm{Lin}(\{f\})$ is one-dimensional. $S^\perp$ is the orthogonal complement of a subset $S \subset H$ and $\mathbb{C}S = \{cs : c \in \mathbb{C}, s \in S\}$ the set of complex multiples of $s \in S$. B^* denotes the adjoint and $\mathrm{dom}(B)$ the maximal domain of an operator B.

1.1 From Classical to Quantum Systems

First we review some of the (static) aspects of quantum theory, which are useful to our considerations and how they relate to the classical (Kolmogorovian) model of probability. A similar outline can be found in [13]. For a more detailed exposition on the structure of quantum theory see [47] or [48]. An excellent introduction to Hilbert space is [4].

To describe a quantum system the following objects are used:

1. a Hilbert space H, the normalised elements of which are called wave functions,

2. the algebra $\mathcal{B}(H)$ of bounded linear operators on H, the self-adjoint ones being known as observables, and

3. a positive linear functional τ on $\mathcal{B}(H)$ which is normalised $\big(\tau(\mathbb{1}_H) = 1\big)$, called a state.

A state τ is termed normal, if there exists a positive trace-class operator ϱ such that

$$\tau(B) = \tau_\varrho(B) = \operatorname{tr}(\varrho B) \qquad \big(B \in \mathcal{B}(H)\big). \tag{1.1.1}$$

Thereby

$$\operatorname{tr}(C) = \sum_{n\in\mathbb{N}} \langle e_n, Ce_n\rangle \qquad \big(C \in \mathcal{B}(H)\big), \tag{1.1.2}$$

denotes the trace of C, i.e. the sum of the diagonal elements of a matrix-representation of C, which is independent of the chosen orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H. ϱ is called the density matrix of τ.

For example, each wave function (i.e. normalised) $f \in H$ defines a normal state τ_f via

$$\tau_f(B) = \langle f, Bf\rangle = \operatorname{tr}(\varrho_f B) \qquad \big(B \in \mathcal{B}(H)\big), \tag{1.1.3}$$

where the density matrix ϱ_f is the projection onto the subspace of H spanned by f, i.e.

$$\varrho_f g = \operatorname{Proj}_f g = \langle f, g\rangle\, f \qquad (g \in H). \tag{1.1.4}$$

Such a state is said to be pure. f itself is also sometimes referred to as a pure state. By normalisation

$$\widehat{f} := f/\left\|f\right\| \qquad (0 \neq f \in H), \tag{1.1.5}$$

every non-zero $f \in H$ determines a pure state $\tau_{\widehat{f}}$.

For a (finite or infinite) probability sequence $(a_n)_{n=1}^N$, i.e. $a_n \geq 0, \sum_{n=1}^N a_n = 1, N \leq \infty$, the mixture of the pure states τ_{f_n} is defined through

$$\tau(B) = \sum_{n=1}^N a_n \tau_{f_n}(B) = \sum_{n=1}^N a_n \langle f_n, Bf_n\rangle \qquad \big(B \in \mathcal{B}(H)\big). \tag{1.1.6}$$

It is again a normal state. In fact, every normal state has such a representation. Hence, normal states which are not pure are referred to as mixed. In addition, the f_n may be chosen mutually orthogonal, i.e. they constitute an orthonormal system.

So, how to recover classical probability from these objects. Henceforth, assume $H = L^2(G)$ for some measure space (G, μ) and define $O_g : \operatorname{dom}(O_g) \to H$, the operator of multiplication with $g \in \mathbb{M}(G)$, through

$$(O_g f)(x) := (g \cdot f)(x) := g(x) \cdot f(x) \qquad \big(f \in \operatorname{dom}(O_g)\,, x \in G\big). \tag{1.1.7}$$

Observe that $O_g \in \mathcal{B}(H)$ if and only if g is essentially bounded, i.e. there exists bounded $h \in \mathbb{M}(G)$ such that $g = h \quad \mu$-almost everywhere. In particular, $O_A := O_{\chi_A} \in \mathcal{B}(H)$ for all $A \in \mathfrak{G}$ and, for a wave function f,

$$Q_{\tau_f}(A) := \tau_f(O_A) = \langle f, O_A f\rangle = \int_A |f|^2\, d\mu \qquad (A \in \mathfrak{G}) \tag{1.1.8}$$

defines a probability measure on G, called the position distribution of the state τ_f. It is absolutely continuous with respect to μ having Radon-Nikodym derivative $\frac{dQ_f}{d\mu} = |f|^2$, called the probability amplitude of f. If μ is a probability measure we have $\mu = Q_{\tau_g}$ for all $g \in \mathbb{M}(G)$ such that $|g|^2 \equiv 1$, i.e. $g(x) = e^{it(x)}$ for some real-valued t and μ-a.e. $x \in G$.

From (1.1.6) and (1.1.8) it is also seen, that every normal state $\tau = \sum_{n=1}^{N} a_n \tau_{f_n}$ has a position distribution given by

$$Q_\tau(A) := \tau(O_A) = \sum_{n=1}^{N} a_n Q_{f_n} = \int_A \sum_{n=1}^{N} a_n |f_n|^2 \, d\mu \qquad (A \in \mathfrak{G}), \tag{1.1.9}$$

i.e. Q_τ is absolutely continuous with respect to μ with Radon-Nikodym derivative $\frac{dQ_\tau}{d\mu} = \sum_{n=1}^{N} a_n |f_n|^2$.

Remark 1.1.1 *Observe that if the wave functions f_n are mutually orthogonal, their superposition $f = \sum_{n=1}^{N} \sqrt{a_n} f_n$ defines another pure state with the same position distribution as the mixed state $\tau = \sum_{n=1}^{N} a_n \tau_{f_n}$, even though these two states are quite different, if the whole quantum context is being considered. We have*

$$Q_\tau = Q_{\tau_f}, \quad \textit{even though} \quad \tau \neq \tau_f. \tag{1.1.10}$$

For bounded $Z \in \mathbb{M}(G, \mathbb{R})$ and a normal state τ it is seen from (1.1.6) and (1.1.9) that

$$\tau(O_Z) = \int Z \, dQ_\tau = \mathbb{E}Z, \tag{1.1.11}$$

if Z is considered as a random variable on $(G, \mathfrak{G}, Q_\tau)$. Hence, the multiplication operators O_Z are quantum representations of random variables Z, interpreted as the measurement of Z on the quantum system in the state τ. As a generalisation of (1.1.11), $\tau(B)$ is called the quantum mechanical expectation of the observable $B \in \mathcal{B}(H)$.

We are interested in the description of quantum point systems. But first we will look at them from a classical point of view. This leads to the theory of point processes.

1.2 Point Processes

Mainly following [62] and [7], this section introduces the basic notions and ideas of the theory of point processes: random configurations of points in space. In section 1.3 we will then add the necessary quantum flavour introduced in section 1.1 and see how point processes relate to the so-called position distribution of quantum, especially coherent, states.

We will model quantum point systems with points in a measure space $(G, \mathfrak{G})$, consisting of the complete separable metric space G and the corresponding σ-algebra of Borel sets $\mathfrak{G}$. For example, G may be chosen to be $\mathbb{R}^k$.

Let $M(G)$ be the set of locally finite counting measures on $(G, \mathfrak{G})$, i.e.

$$M(G) := \{\varphi : \varphi \text{ measure on } (G, \mathfrak{G}), \varphi(A) \in \mathbb{N}_0 \text{ for all bounded } A \in \mathfrak{G}\}. \tag{1.2.1}$$

Denote with $\mathcal{O}$ the zero measure in $M(G)$, i.e. $\mathcal{O}(G) = 0$, and by δ_x the Dirac measure concentrated in x. Then each $\mathcal{O} \neq \varphi \in M(G)$ has a unique representation $\varphi = \sum_{j\in J} \delta_{x_j}$ with an at most countable index set J and the sequence $(x_j)_{j\in J}$ in G having no accumulation points. Hence, the elements of $M(G)$ may be interpreted as locally finite, symmetric point configurations in G, $\mathcal{O}$ representing the empty configuration. For two configurations $\varphi, \hat{\varphi}$ we will write $\hat{\varphi} \leq \varphi$ in case $\hat{\varphi}(A) \leq \varphi(A)$ for all $A \in \mathfrak{G}$, which means that $\hat{\varphi}$ is a subconfiguration of φ. Also, $x \in \varphi$ stands for $\delta_x \leq \varphi$, i.e. $\varphi(\{x\}) > 0$.

Define

$$M_n(G) := \{\varphi \in M(G) : \varphi(G) = n\} \qquad (n \in \mathbb{N}_0) \tag{1.2.2}$$

to be the subset of configurations having exactly n points and also give $M_{\geq n}(G)$, $M_{<n}(G)$, $M_{<\infty}(G), \dots$ their obvious meaning. Thereby, $M_{<\infty}(G)$ is called the set of finite point configurations. The set of so-called simple counting measures or simple configurations is denoted with

$$M^s(G) := \{\varphi \in M(G) : \varphi(\{x\}) \leq 1 \text{ for all } x \in G\}. \tag{1.2.3}$$

Finally, equip $M(G)$ with its canonical σ-algebra $\mathfrak{M}(G)$, that is, the smallest σ-algebra containing all sets of the form $\{\varphi \in M(G) : \varphi(A) = k\}$ for bounded $A \in \mathfrak{G}$ and $k \in \mathbb{N}_0$. Observe that $\mathfrak{M}(G)$ is also the smallest σ-algebra making all the evaluation maps $\varphi \mapsto \varphi(A)$ measurable for all bounded $A \in \mathfrak{G}$ and that all of the sets defined above, in particular $M_n(G), M_{<\infty}(G)$ and $M^s(G)$ belong to $\mathfrak{M}(G)$.

Definition 1.2.1 *Let $(\Omega, \mathcal{F}, P)$ be a probability space. A measurable map $X : \Omega \to M(G)$, i.e. an $M(G)$-valued random variable, is called a point process. Its distribution law is denoted by $P_X := P \circ X^{-1}$. A propability measure on $\big(M(G), \mathfrak{M}(G)\big)$ is also sometimes called a (canonical) point process. A point process X is called simple/finite/etc. in case its realisations $X(\omega)$ have this property almost surely.*

Recalling

$$\int_G f\, d\varphi = \sum_{x\in\varphi} f(x) \qquad \big(f \in \mathbb{M}(G), \varphi \in M(G)\big), \tag{1.2.4}$$

whenever these integrals exist, for a point process X we set

$$X_f(\omega) := \int_G f\, d\big(X(\omega)\big) = \sum_{x\in X(\omega)} f(x) \qquad \big(f \in \mathbb{M}(G), \omega \in \Omega\big) \tag{1.2.5}$$

to be the random total measurement of f at points given by X. In particular, $X_A : \Omega \to \mathbb{N}_0 \cup \{\infty\}$ is defined through $X_A := X_{\chi_A}$, i.e.

$$X_A(\omega) = \sum_{x \in X(\omega)} \chi_A(x) = [X(\omega)](A) \qquad (A \in \mathfrak{G}, \omega \in \Omega), \tag{1.2.6}$$

to be the random number of points of the point process X in A. We have the following characterisation:

Proposition 1.2.2 *A map $X : \Omega \to M(G)$ is a point process if and only if X_A defined as in* (1.2.6) *is an $\mathbb{N}_0 \cup \{\infty\}$-valued random number for all $A \in \mathfrak{G}$.*

This justifies identifying a point process X with the random field (X_A):

$$X = (X_A)_{A \in \mathfrak{G}}. \tag{1.2.7}$$

Remark 1.2.3 *For a point process X and $0 \leq f \in \mathbb{M}(G)$, X_f is also an (extended) random number: f can be approximated by simple functions, allowing approximation of X_f by X_A's through monotone convergence.*

The intensity functional of the point process X is given by

$$\mu_X(f) := \mathbb{E}X_f = \int\limits_\Omega X_f \, dP = \int\limits_{M(G)} \int\limits_G f \, d\varphi \, dP_X(\varphi) \qquad \big(f \in \mathbb{M}(G)\big), \tag{1.2.8}$$

if this integral exists. It induces a measure μ_X on $\mathfrak{G}$ defined through

$$\mu_X(A) := \mu_X(\chi_A) = \mathbb{E}X_A \qquad (A \in \mathfrak{G}), \tag{1.2.9}$$

the so-called intensity measure of X. Assuming existence of the integrals, approximation of measurable functions with simple ones also yields the following relation between intensity functional and measure:

$$\mu_X(f) = \int_G f \, d\mu_X \qquad \big(f \in \mathbb{M}(G)\big). \tag{1.2.10}$$

There is a vast number of tools to characterise the distribution of a point process. A basic selection of them will be given in the following

Proposition 1.2.4 *The distribution of a point process X is completely determined by both of the following:*

1. *Its Laplace functional L_X given by*

$$L_X(f) := \mathbb{E}e^{-X_f} \qquad \big(\textit{bounded, non-negative } f \in \mathbb{M}(G)\big). \tag{1.2.11}$$

2. *The finite-dimensional distributions*

$$P_{X_{A_1},\ldots,X_{A_n}}(k_1,\ldots,k_n) := P\left(X_{A_1} = k_1,\ldots,X_{A_n} = k_n\right)$$
$$\big(n \in \mathbb{N}, k_1,\ldots,k_n \in \mathbb{N}_0, A_1,\ldots,A_n \in \mathfrak{G}\big). \tag{1.2.12}$$

   Thereby, it is enough to consider bounded $A_1,\ldots,A_n \in \mathfrak{G}$ for uniqueness.

Restricting the finite-dimensional distributions to the case $n = 1$ and $k = 0$ defines the so-called avoidance (or vacuity) function

$$P_X^0(A) := P_{X_A}(0) = P\left(X_A = 0\right) \qquad (A \in \mathfrak{G}) \tag{1.2.13}$$

of the point process X. Remarkably enough, for simple point processes we have

Proposition 1.2.5 *The distribution of a simple point process X is completely determined by its avoidance function P_X^0.*

The most important example of a point process is the Poisson process.

Definition 1.2.6 *Let Λ be a locally finite measure on G. If a point process X satisfies*

1. *X_A has a Poisson distribution with mean $\Lambda(A)$ for all bounded $A \in \mathfrak{G}$.*
2. *The random variables $X_{A_1},\ldots,X_{A_n}$ are independent for all $n \in \mathbb{N}$ and disjoint $A_1,\ldots,A_n \in \mathfrak{G}$.*

it is called Poisson process with intensity measure Λ.

As the name suggests, $\mu_X := \Lambda$ is the intensity measure of X as defined in (1.2.9). To interpret a Poisson process we will take a closer look at how it may be constructed for finite Λ. In this case $\Lambda^* := \frac{\Lambda}{\Lambda(G)}$ is a probability measure on G. Now assume $X_1, X_2, \ldots$ are independent G-valued random variables each distributed according to Λ^* and N a Poisson random variable with mean $\Lambda(G)$ independent of the X_i. Set

$$X(\omega) := \sum_{k=1}^{N(\omega)} \delta_{X_k(\omega)} \qquad (\omega \in \Omega), \tag{1.2.14}$$

where this is supposed to mean $X(\omega) = \mathcal{O}$ for all $\omega \in \Omega$ such that $N(\omega) = 0$.

Proposition 1.2.7 (see [62, Proposition 3.6]) *X defined in (1.2.14) is a Poisson process with intensity measure Λ.* ■

We therefore have the following interpretation: A Poisson process with intensity measure Λ describes a Poisson random number of independent particles each having the same position distribution Λ^*. The intensity of the random point configuration (mean number of points) is given by $\Lambda(G)$.

Remark 1.2.8 *By decomposing G into disjoint and bounded regions, a similar construction, hence interpretation, may be given in case Λ is only locally finite.*

Using Propositions 1.2.4 and 1.2.5, a Poisson process is characterised by

Proposition 1.2.9 *Let Λ be a locally finite measure on G and X a point process. Then X is a Poisson process with intensity measure Λ if and only if its Laplace functional is given by*

$$L_X(f) = e^{\int (e^{-f}-1)\, d\Lambda} \qquad \big(\text{bounded, non-negative } f \in \mathbb{M}(G)\big). \tag{1.2.15}$$

In case X is even simple, it is a Poisson process with intensity measure Λ if and only if its avoidance function satisfies

$$P_X^0(A) = P\left(X_A = 0\right) = e^{-\Lambda(A)} \qquad \big(A \in \mathfrak{G}\big). \tag{1.2.16}$$

1.3 The Symmetric Fock Space

We are now ready to add some quantum flavour introduced in section 1.1 to the theory of point processes from section 1.2: a quantisation of point processes. Thereby, we follow the ideas introduced in [16, 17, 18]. A similar approach was taken in [44, 55]. First we equip our location space $(G, \mathfrak{G})$ with a locally finite, diffuse (non-atomic) measure μ, i.e. $\mu(A) < \infty$ for bounded $A \in \mathfrak{G}$ and $\mu(\{x\}) = 0$ for all $x \in G$ (see remark 1.3.5 for not necessarily non-atomic measures). For example, G may be chosen to be $\mathbb{R}^k$ and μ Lebesque measure. This measure μ induces a measure on the space of configurations $\big(M(G), \mathfrak{M}(G)\big)$, called Fock space measure, defined through

$$F_\mu = \delta_{\mathcal{O}} + \sum_{n=1}^{\infty} \frac{\mu^{\otimes n} \circ s_n^{-1}}{n!}, \tag{1.3.1}$$

where $s_n : G^n \to M(G)$ with

$$s_n(x_1, \ldots, x_n) := \sum_{k=1}^{n} \delta_{x_i}$$

denotes the symmetric embedding of G^n into $M(G)$ and $\mu^{\otimes n} \circ s_n^{-1}$ the image of the n-fold product measure $\mu^{\otimes n}$ of μ under the measurable map s_n.

Remark 1.3.1 *That s_n is indeed measurable was shown in [56].*

Remark 1.3.2 *As $n!$ is the number of permutations of n elements, we see by (1.3.1) that the restriction of F_μ to n-point configurations is the usual product measure $\mu^{\otimes n}$, but symmetrised. Therefore F_μ is concentrated on the set $M^s(G) \cap M_{<\infty}(G)$ of both simple and finite configurations. If μ is even finite so is F_μ and we have*

$$F_\mu\big(M(G)\big) = \sum_{n=0}^{\infty} F_\mu\big(M_n(G)\big) = \sum_{n=0}^{\infty} \frac{\big(\mu(G)\big)^n}{n!} = e^{\mu(G)}.$$

By $\mathcal{M}(G)$ we denote the space of (equivalence classes of) square integrable, measurable, complex-valued functions on $M(G)$, i. e.

$$\begin{aligned}\mathcal{M}(G) &:= L^2\big(M(G)\big) = L^2\big(M(G), \mathfrak{M}(G), F_\mu\big) \\ &= \left\{ \Psi \in \mathbb{M}(G) : \|\Psi\|^2 := \int_{M(G)} |\Psi(\varphi)|^2 \, dF_\mu(\varphi) < \infty \right\} \end{aligned} \tag{1.3.2}$$

with scalar product

$$\langle \Psi, \Phi \rangle := \int_{M(G)} \overline{\Psi(\varphi)} \cdot \Phi(\varphi) \, dF_\mu(\varphi) \qquad \big(\Psi, \Phi \in \mathcal{M}(G)\big), \tag{1.3.3}$$

where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$.

Definition 1.3.3 *$\mathcal{M}(G)$ is called the Boson (or symmetric) Fock space over $L^2(G)$.*

Since this work only deals with symmetric Fock space, $\mathcal{M}(G)$ will simply be referred to as Fock space.

Remark 1.3.4 *Usually, the symmetric Fock space over a separable Hilbert space H is defined as*

$$\Gamma(H) := \bigoplus_{n=0}^{\infty} H_{\mathrm{sym}}^{\otimes n}, \tag{1.3.4}$$

where $H_{\mathrm{sym}}^{\otimes n}$ denotes the symmetrised n-fold tensor product of H [58, 60]. This means, that $\Psi \in \Gamma(H)$ is a sequence $\Psi = (\Psi_n)_{n \in \mathbb{N}_0}$ with $\Psi_0 \in H_{\mathrm{sym}}^{\otimes 0} := \mathbb{C}$ and each $\Psi_n \in H_{\mathrm{sym}}^{\otimes n}$. The scalar product is given by

$$\begin{aligned}\langle \Psi, \Phi \rangle_{\Gamma(H)} = \overline{\Psi_0}\Phi_0 + \sum_{n \in \mathbb{N}} \frac{1}{n!} \cdot \langle \Psi_n, \Phi_n \rangle_{H^{\otimes n}} \\ \big(\Psi = (\Psi_n)_{n \in \mathbb{N}_0}, \Phi = (\Phi_n)_{n \in \mathbb{N}_0} \in \Gamma(H)\big).\end{aligned} \tag{1.3.5}$$

It was shown in [35] that, for $H := L^2(G)$ and all $n \in \mathbb{N}_0$, $L^2\big(M_n(G)\big)$ and $H_{\mathrm{sym}}^{\otimes n}$, hence $\mathcal{M}(G)$ and $\Gamma\big(L^2(G)\big)$ are isomorphic, because

$$\mathcal{M}(G) = L^2\big(M(G)\big) = L^2\left(\bigcup_{n=0}^{\infty} M_n(G)\right) = \bigoplus_{n=0}^{\infty} L^2\big(M_n(G)\big).$$

Remark 1.3.5 *Treatment of symmetric Fock space in the language of point processes for atomic or general (not necessarily non-atomic) measures can be found in [8, 51, 63]. It should be noted, though, that most of this chapter, except for proposition 1.3.6 and the proof of proposition 1.3.8 where simplicity is used, remain valid in the general case. This is in particular true for the sections on exponential vectors (1.4) and decomposition of Fock space (1.5 and 1.6), but not for the results in chapters 2 and 3.*

Symmetric quantum particle configurations located in G are represented by states on the symmetric Fock space $\mathcal{M}(G)$. According to (1.1.6) each normal state τ is a mixture of pure states given by

$$\tau(B) = \sum_{n=1}^{N} a_n \tau_{\Phi_n}(B) = \sum_{n=1}^{N} a_n \langle f_n, B f_n\rangle \qquad \Big(B \in \mathcal{B}\big(\mathcal{M}(G)\big)\Big) \tag{1.3.6}$$

for an orthonormal system of wave functions $\Phi_n \in \mathcal{M}(G)$ and a probability sequence $(a_n)_{n=1}^{N}$, $N \le \infty$.

Operators of multiplication $O_\Psi : \mathrm{dom}(O_\Psi) \to \mathcal{M}(G)$ are now given for $\Psi \in \mathbb{M}\big(M(G)\big)$ through

$$(O_\Psi \Phi)(\varphi) := (\Psi \cdot \Phi)(\varphi) := \Psi(\varphi) \cdot \Phi(\varphi) \qquad \big(\Phi \in \mathrm{dom}(O_\Psi)\,, \varphi \in M(G)\big). \tag{1.3.7}$$

In particular, $O_Y := O_{\chi_Y}$ is bounded for all measurable sets of configurations $Y \in \mathfrak{M}(G)$ and, as in (1.1.9),

$$Q_\tau(Y) := \tau(O_Y) = \int_Y \sum_{n=1}^{N} a_n \left|\Phi_n\right|^2 \, dF_\mu \qquad \big(Y \in \mathfrak{M}(G)\big), \tag{1.3.8}$$

defines a probability measure on $M(G)$, i.e. the distribution of a point process called the position distribution of the normal state τ. It is absolutely continuous with respect to the Fock space measure F_μ with Radon-Nikodym derivative $\frac{dQ_\tau}{dF_\mu} = \sum_{n=1}^{N} a_n \left|\Phi_n\right|^2$. As F_μ is concentrated on the simple and finite configurations, a point process X with distribution Q_τ is also both simple and finite. Hence, by Proposition 1.2.5, Q_τ is completely determined by its avoidance function Q_τ^0. This means that, in (1.3.8), it is enough to consider Y's of the form $Y = \{\varphi \in M(G) : \varphi(A) = 0\}$ for $A \in \mathfrak{G}$ because

$$Q_\tau^0(A) = Q_\tau\big(\{\varphi \in M(G) : \varphi(A) = 0\}\big) \qquad (A \in \mathfrak{G}). \tag{1.3.9}$$

We have therefore shown

Proposition 1.3.6 *The position distribution Q_τ of a normal state τ on the symmetric Fock space $\mathcal{M}(G)$ is completely determined by its avoidance function Q^0_τ.* ■

We will now introduce the quantum analogues of the Poisson processes. They are given by a particular kind of pure states, called coherent.

For $f : G \to \mathbb{C}$ denote with

$$\exp[f]\,(\varphi) := \begin{cases} 1 & \text{if } \varphi = \mathcal{O} \\ 0 & \text{if } \varphi(G) = \infty \\ \prod\limits_{x \in \varphi} f(x) & \text{if } 0 < \varphi(G) < \infty \end{cases} \qquad \big(\varphi \in M(G)\big) \tag{1.3.10}$$

the coherent function generated by f. It is well known that $\exp[f] \in \mathcal{M}(G)$ if and only if $f \in L^2(G)$. In this case $\exp[f]$ is called exponential vector and we have

$$\|\exp[f]\|^2_{\mathcal{M}(G)} = e^{\|f\|^2_{L^2(G)}}. \tag{1.3.11}$$

For a thorough exposition on exponential vectors see section 1.4.

Definition 1.3.7 *A pure state with wave function given by a normalised exponential vector $\widehat{\exp[f]} = e^{-\frac{\|f\|^2}{2}} \exp[f]$ for some $f \in L^2(G)$ is called coherent.*

Proposition 1.3.8 *The position distribution of a coherent state with wave function $\widehat{\exp[f]}$ is a Poisson process with intensity measure $\Lambda = |f|^2\,\mu$, i.e.*

$$\Lambda(A) = \int_A |f|^2 \, d\mu \qquad (A \in \mathfrak{G}).$$

Proof: Using (1.3.8) and (1.3.11) we have for all $A \in \mathfrak{G}$

$$\begin{aligned} Q^0_{\tau_{\widehat{\exp[f]}}}(A) &= Q_{\tau_{\widehat{\exp[f]}}}\big(\{\varphi \in M(G) : \varphi(A) = 0\}\big) \\ &= \int\limits_{\{\varphi \in M(G) : \varphi(A) = 0\}} \left|\widehat{\exp[f]}\right|^2 dF_\mu \\ &= e^{-\|f\|^2} \int\limits_{\{\varphi \in M(G) : \varphi(A) = 0\}} |\exp[f]|^2 \, dF_\mu \\ &= e^{-\|f\|^2} \int |\exp[f \cdot \chi_{A^c}]|^2 \, dF_\mu = e^{-\|f\|^2} \, \|\exp[f \cdot \chi_{A^c}]\|^2 \\ &= e^{-\|f\|^2} e^{\|f \cdot \chi_{A^c}\|^2} = e^{-\|f \cdot \chi_A\|^2} = e^{-\int_A |f|^2 \, d\mu} = e^{-\Lambda(A)}. \end{aligned}$$

Since this is the vacuity function of a Poisson process (see 1.2.9), which determines the position distribution of a normal state by Proposition 1.3.6, the proof is complete. ■

As a generalisation of the well-known characterisation of Poisson processes by the property of so-called local independence (see [56]), it was shown in the more general setting of locally normal states in [32] that coherent states are characterised by (quantum) local independence.

1.4 Exponential Vectors

A very important class of functions from Fock space is the set of exponential vectors. Their importance mainly rests on the fact that they are linearly independent and total (i.e. their linear span is dense). Thus they may be used to define linear operators or to find isometric identifications of Fock spaces. Because working with exponential vectors lies at the core of this thesis, quite a few of the required properties and results concerning them will be presented including their proofs, even though most of them may already be found in the literature.

For $f : G \to \mathbb{C}$ recall

$$\exp[f]\,(\varphi) := \begin{cases} 1 & \text{if } \varphi = \mathcal{O} \\ 0 & \text{if } \varphi(G) = \infty \\ \prod\limits_{x \in \varphi} f(x) & \text{if } 0 < \varphi(G) < \infty \end{cases} \qquad \big(\varphi \in M(G)\big) \tag{1.4.1}$$

to be the coherent function generated by f. It is well known that $\exp[f] \in \mathcal{M}(G)$ if and only if $f \in L^2(G)$. In this case $\exp[f]$ is called exponential vector. The exponential vector $\exp[0]$ is called vacuum.

Remark 1.4.1 *For $f \in H$, the exponential vector $\exp[f] \in \Gamma(H)$ is given by the sequence $\exp[f] = (f^{\otimes n})_{n \in \mathbb{N}_0} = (1, f, f^{\otimes 2}, \dots)$.*

The following properties of coherent functions will be needed subsequently. They are all well-known (see [51] for instance), except for (1.4.7). Thereby (1.4.6) is a generalisation of the Binomial Theorem, the usual one being recovered for constant f, g and finite φ.

Lemma 1.4.2 *If $f, g : G \to \mathbb{C}$ and $\varphi, \varphi' \in M\big(G\big)$, then*

$$|\exp[f]\,(\varphi)| = \exp[|f|]\,(\varphi), \tag{1.4.2}$$

$$\overline{\exp[f]\,(\varphi)} = \exp\big[\overline{f}\big]\,(\varphi) \tag{1.4.3}$$

$$\exp[f]\,(\varphi + \varphi') = \exp[f]\,(\varphi) \cdot \exp[f]\,(\varphi'), \tag{1.4.4}$$

$$\exp[f \cdot g]\,(\varphi) = \exp[f]\,(\varphi) \cdot \exp[g]\,(\varphi). \tag{1.4.5}$$

In case φ is simple, we also have

$$\exp[f+g]\,(\varphi) = \sum_{\varphi_1+\varphi_2=\varphi} \exp[f]\,(\varphi_1)\cdot \exp[g]\,(\varphi_2) \tag{1.4.6}$$

and

$$|\exp[f]\,(\varphi) + \exp[g]\,(\varphi)| \le \exp[|f|+|g|]\,(\varphi) \qquad (\varphi \ne \mathcal{O}). \tag{1.4.7}$$

If, in addition, f if μ-integrable or non-negative, then

$$\int_{M_n(G)} \exp[f]\ dF_\mu = \frac{\left(\int_G f\,d\mu\right)^n}{n!} \qquad (n \in \mathbb{N}_0) \tag{1.4.8}$$

and

$$\int_{M(G)} \exp[f]\ dF_\mu = e^{\int_G f\,d\mu}. \tag{1.4.9}$$

Finally, $\exp[f] \in \mathcal{M}(G)$ *if and only if* $f \in L^2(G)$ *and*

$$\|\exp[f]\|^2_{\mathcal{M}(G)} = e^{\|f\|^2_{L^2(G)}} \qquad \text{and} \qquad \langle \exp[f]\,,\exp[g]\rangle_{\mathcal{M}(G)} = e^{\langle f,g\rangle_{L^2(G)}}$$
$$\big(f,g \in L^2(G)\big). \tag{1.4.10}$$

Proof: (1.4.2), (1.4.3), (1.4.4) and (1.4.5) are immediate from definition (equation (1.4.1)).

We will now show (1.4.6). For $\varphi(G) = \infty$, both sides vanish because in this case $\varphi_1(G) = \infty$ or $\varphi_2(G) = \infty$ and hence $\exp[f]\,(\varphi_1)\cdot\exp[g]\,(\varphi_2) = 0$. For $\varphi(G) \in \mathbb{N}_0$ we will use induction. If $\varphi = \mathcal{O}$, both sides of the equation are equal to 1 because there is only one summand ($\varphi_1 = \varphi_2 = \mathcal{O}$). Now assume (1.4.6) holds for all simple $\varphi \in M_{<n}(G)$ for some $n \in \mathbb{N}_0$. We are to show that it also holds for $\varphi + \delta_x$ for all $x \in G, x \notin \varphi$. Using (1.4.1) and the induction hypothesis we have

$$\begin{aligned}
\exp[f+g]\,(\varphi+\delta_x) &= \big(f(x)+g(x)\big)\exp[f+g]\,(\varphi) \\
&= \sum_{\varphi_1+\varphi_2=\varphi} \exp[f]\,(\varphi_1+\delta_x)\cdot\exp[g]\,(\varphi_2) \\
&\quad + \sum_{\varphi_1+\varphi_2=\varphi} \exp[f]\,(\varphi_1)\cdot\exp[g]\,(\varphi_2+\delta_x) \\
&= \sum_{\substack{\varphi_1+\varphi_2=\varphi+\delta_x \\ x\in\varphi_1}} \exp[f]\,(\varphi_1)\cdot\exp[g]\,(\varphi_2) \\
&\quad + \sum_{\substack{\varphi_1+\varphi_2=\varphi+\delta_x \\ x\in\varphi_2}} \exp[f]\,(\varphi_1)\cdot\exp[g]\,(\varphi_2) \\
&= \sum_{\varphi_1+\varphi_2=\varphi+\delta_x} \exp[f]\,(\varphi_1)\cdot\exp[g]\,(\varphi_2).
\end{aligned}$$

(1.4.7) follows from the triangular inequality, (1.4.2) and (1.4.6):

$$\begin{aligned}
&|\exp[f]\,(\varphi) + \exp[g]\,(\varphi)| \\
&\qquad \leq \ |\exp[f]\,(\varphi)| + |\exp[g]\,(\varphi)| \\
&\qquad = \ \exp[|f|]\,(\varphi) + \exp[|g|]\,(\varphi) \\
&\qquad = \ \exp[|f|]\,(\varphi) \cdot \exp[|g|]\,(\mathcal{O}) + \exp[|f|]\,(\mathcal{O}) \cdot \exp[|g|]\,(\varphi) \\
&\qquad \leq \ \sum_{\varphi_1+\varphi_2=\varphi} \exp[|f|]\,(\varphi_1) \cdot \exp[|g|]\,(\varphi_2) \\
&\qquad = \ \exp[|f| + |g|]\,(\varphi) \qquad (\text{simple } \varphi \neq \mathcal{O}).
\end{aligned}$$

By (1.3.1) (definition of F_μ), we have for integrable f

$$\int_{M_0(G)} \exp[f]\ dF_\mu = \exp[f]\,(\mathcal{O}) = 1$$

and for all $n \in \mathbb{N}$

$$\begin{aligned}
\int_{M_n(G)} \exp[f]\ dF_\mu &= \frac{1}{n!} \int_{G^n} \exp[f] \left(\sum_{k=1}^{n} \delta_{x_k} \right) d\mu^{\otimes n}(x_1, \ldots, x_n) \\
&= \frac{1}{n!} \int_{G^n} \prod_{k=1}^{n} f(x_k)\, d\mu^{\otimes n}(x_1, \ldots, x_n) = \frac{1}{n!} \left(\int_G f\, d\mu \right)^n,
\end{aligned}$$

showing (1.4.8). (1.4.9) is immediate from (1.4.8) and

$$\int_{M(G)} \exp[f]\ dF_\mu = \sum_{n\in\mathbb{N}_0} \int_{M_n(G)} \exp[f]\ dF_\mu.$$

For integrable f, g, (1.4.3) and (1.4.5) imply $\overline{\exp[f]} \cdot \exp[g] = \exp\left[\overline{f} \cdot g\right]$ and therefore, using (1.4.9),

$$\begin{aligned}
\langle \exp[f]\,, \exp[g] \rangle_{\mathcal{M}(G)} &= \int_{M(G)} \overline{\exp[f]} \cdot \exp[g]\ dF_\mu \\
&= \int_{M(G)} \exp\left[\overline{f} \cdot g\right]\ dF_\mu = e^{\int_G \overline{f} \cdot g\, d\mu} = e^{\langle f,g \rangle_{L^2(G)}},
\end{aligned}$$

which proves (1.4.10). ■

Remark 1.4.3 *For $f \in H$ equations* (1.4.2) *through* (1.4.9) *of Lemma 1.4.2 are meaningsless but* (1.4.10) *holds respectively, i.e.*

$$\|\exp[f]\|^2_{\Gamma(H)} = e^{\|f\|^2_H} \quad \text{and} \quad \langle \exp[f]\,, \exp[g] \rangle_{\Gamma(H)} = e^{\langle f,g \rangle_H} \qquad (f, g \in H). \tag{1.4.11}$$

In the sequel, definitions and properties in $\Gamma(H)$, that do not depend on its representation as $\Gamma(H) \cong \mathcal{M}(G) = L^2\big(M(G)\big)$ for some G such that $H = L^2(G)$, will be formulated in the $\Gamma(H)$-notation.

Lemma 1.4.4 *The map $f \mapsto \exp[f]$ from H to $\Gamma(H)$ is one-to-one and continuous.*

Proof: The first part is a consequence of f being the projection of $\exp[f]$ onto the one-particle subspace $H = H_{\text{sym}}^{\otimes 1} \subset \Gamma(H)$, i.e. $f = \mathrm{Proj}_H \exp[f]$. Continutity follows from continutity of the scalar product and

$$\|\exp[f] - \exp[g]\|_{\Gamma(H)}^2 = e^{\|f\|_H^2} + e^{\|g\|_H^2} - 2\mathrm{Re}\, e^{\langle f,g\rangle_H} \qquad \big(f, g \in H\big).$$

■

Denote with $\exp[H] := \{\exp[f] : f \in H\}$ the set of all exponential vectors from a Hilbert space H and with $\mathcal{E}(H) := \mathrm{Lin}(\exp[H])$ the linear space generated by the exponential vectors, called the exponential domain.

Proposition 1.4.5 (see [60, Proposition 19.4] and [28, Proposition 2.3.2]) *The set $\exp[H]$ is linearly independent and total in $\Gamma(H)$.*

Proof: Fix arbitrary $n \in \mathbb{N}$ and distinct $f_1, \dots, f_n \in H$ and assume $\sum_{i=1}^{n} \lambda_i \exp[f_i] = 0$ for some $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. Then

$$0 = k! \left\langle f^{\otimes k}, \sum_{i=1}^{n} \lambda_i \exp[f_i] \right\rangle_{\Gamma(H)} = \sum_{i=1}^{n} \lambda_i \langle f, f_i\rangle_H^k \qquad \big(f \in H, k \in \mathbb{N}_0\big), \tag{1.4.12}$$

implying the linear system

$$A_f \lambda = 0 \qquad (f \in H), \tag{1.4.13}$$

with Vandermonde matrix $A_f := \Big(\langle f, f_i\rangle^{k-1}\Big)_{k,i=1}^{n}$ and $\lambda := (\lambda_1, \dots, \lambda_n)$.

For every pair $i \neq j$ the sets $\{f_i - f_j\}^\perp$ ($^\perp$ denotes orthogonal complement) are proper subspaces of H. Using the result from linear algebra that no vector space over an infinite field is a finite union of proper subspaces (see [64, Lemma 1.C.12]),

$$R := \bigcup_{\substack{i,j=1 \\ i\neq j}}^{n} \{f_i - f_j\}^\perp \tag{1.4.14}$$

is a proper subset of H. Choosing $f \in H \setminus R$ implies $\langle f, f_i - f_j\rangle \neq 0$ for $i \neq j$, hence

$$\det A_f = \prod_{\substack{i,j=1 \\ i>j}}^{n} \langle f, f_i - f_j\rangle \neq 0 \tag{1.4.15}$$

and therefore $\lambda = 0$, showing linear independence because n and $f_1, \dots, f_n$ were arbitrary.

To prove totality we are to show that, for $\Psi = (\Psi_n) \in \Gamma(H)$,

$$\langle \Psi, \exp[f] \rangle_{\Gamma(H)} = 0 \qquad \big(f \in H\big) \tag{1.4.16}$$

implies $\Psi = 0$, or equivalently $\Psi_n = 0$ for all $n \in \mathbb{N}_0$.

To this end observe that, because of

$$(tf)^{\otimes n} = t^n f^{\otimes n} \qquad (t \in \mathbb{C}, f \in H, n \in \mathbb{N}_0),$$

where $g^{\otimes 0} := 1$, it is also true that

$$0 = \langle \Psi, \exp[tf] \rangle_{\Gamma(H)} = \sum_{n \in \mathbb{N}_0} \left\langle \Psi_n, (tf)^{\otimes n} \right\rangle_{H_{\mathrm{sym}}^{\otimes n}} = \sum_{n \in \mathbb{N}_0} t^n \left\langle \Psi_n, f^{\otimes n} \right\rangle_{H_{\mathrm{sym}}^{\otimes n}} \qquad (t \in \mathbb{C}, f \in H). \tag{1.4.17}$$

For fixed $f \in H$ the right-hand side of (1.4.17) is a power series in t that vanishes everywhere. Therefore

$$0 = \left\langle \Psi_n, f^{\otimes n} \right\rangle_{H_{\mathrm{sym}}^{\otimes n}} \qquad (n \in \mathbb{N}_0, f \in H). \tag{1.4.18}$$

As $\{f^{\otimes n} : f \in H\}$ is total in $H_{\mathrm{sym}}^{\otimes n}$ for all $n \in \mathbb{N}_0$ (see [58]), the proof is complete. ■

We are now ready to define linear operators by their restriction to exponential vectors (see [60, Corollary 19.5]). For example:

Definition 1.4.6 *Let $T : H \to H'$ be a linear contraction. The unique (again contractive) linear operator $\Gamma(T) : \Gamma(H) \to \Gamma(H')$ defined by extension of*

$$\Gamma(T) \exp[f] := \exp[Tf] \qquad \big(f \in H\big) \tag{1.4.19}$$

is called second quantisation of T. The unitary operator $\mathcal{W}(h) : \Gamma(H) \to \Gamma(H)$ defined through

$$\mathcal{W}(h) \exp[f] := e^{-\frac{1}{2}\|h\|^2 - \langle h, f \rangle} \exp[f + h] \qquad \big(f \in H\big) \tag{1.4.20}$$

is called Weyl operator associated with $h \in H$.

Let us summarise their properties (see [58, subsection IV.1.6] and [60, section 20]):

Lemma 1.4.7 *Let $T : H \to H'$ and $T' : H' \to H''$ be linear contractions and $f, g \in H$. Then*

1. $\Gamma(T')\Gamma(T) = \Gamma(T'T)$.

2. $\Gamma(T^*) = \Gamma(T)^*$.

3. $\Gamma(T)$ *is isometric/unitary if and only if* T *is isometric/unitary.*

4. $\mathbb{1}_{\Gamma(H)} = \Gamma(\mathbb{1}_H) = \mathcal{W}(0)$.

5. $\mathcal{W}(f)\mathcal{W}(g) = e^{-i\,\mathrm{Im}\,\langle f,g\rangle}\mathcal{W}(f+g) = e^{-2i\,\mathrm{Im}\,\langle f,g\rangle}\mathcal{W}(g)\mathcal{W}(f)$.

6. $\mathcal{W}(f)^* = \mathcal{W}(-f)$. ■

We shall see later (see Proposition 2.6.4) that operators of the kind

$$U = c \cdot \mathcal{W}(h)\Gamma(T) \tag{1.4.21}$$

for $c \in \mathbb{C}, |c| = 1, h \in H'$ and isometric $T : H \to H'$ are the only isometries from $\Gamma(H)$ to $\Gamma(H')$ that "preserve multiples of exponential vectors" in the sense that the image of $\exp[H]$ is contained in $\mathbb{C}\exp[H']$.

1.5 Fock space and its Regional Factorisation

Denote with

$$\Sigma(A) \ := \ \{(A_1, \dots, A_n) \ : \ A_1 \cup \cdots \cup A_n \ = \ A, A_1, \dots, A_n \ \in \ \mathfrak{G}, \text{disjoint}\} \tag{1.5.1}$$

the set of all finite measurable decompositions of $A \in \mathfrak{G}$ into disjoint $A_1, \dots, A_n \in \mathfrak{G}$.

Remark 1.5.1 *Although not stated explicitely, we also make the additional assumption of* $\mu(A_k) > 0$ *in* (1.5.1) *for all* $1 \leq k \leq n$*, whenever the issue in question does not make sense otherwise.*

Roughly speaking (ignoring some technical details like associativity and commutativity), a Hilbert space H is called G-factorisable if there exists a family $(H_A)_{A\in\mathfrak{G}}$ of Hilbert spaces such that $H_G \cong H$ and

$$H_A \cong \bigotimes_{k=1}^{n} H_{A_k} \qquad \big((A_1, \dots, A_n) \in \Sigma(A), n \in \mathbb{N}, A \in \mathfrak{G}\big).$$

If the same relations hold with the tensor product being replaced with the orthogonal sum, it is called G-summable.

In some sense, symmetric Fock space is the only factorisable space. This famous result is due to Araki and Woods from 1966 (see [3]). For a finite, non-atomic measure space (G, μ) a Hilbert space $\mathcal{H}$ is G-factorisable if and only if $\mathcal{H} \subseteq \Gamma(H)$ for some G-summable

Hilbert space H. For more details on summable and factorisable Hilbert spaces see [3] and [44].

We will now develop factorisation of symmetric Fock space in more detail, thereby returning to the language of quantised point processes.

If, in the definitions for symmetric Fock space, $(G, \mathfrak{G}, \mu)$ is replaced with $(A, A \cap \mathfrak{G}, \mu_{|A})$, where $A \in \mathfrak{G}$ and $\mu_{|A}$ denotes the restriction of μ to the spur σ-algebra $A \cap \mathfrak{G}$, respective definitions for $M(A), \mathfrak{M}(A), \mathcal{M}(A), \ldots$ are obtained. Hence we find the following identification, which will always be assumed hereafter

$$\mathcal{M}(A) := L^2\big(M(A)\big) \cong \Gamma\big(L^2(A)\big) \qquad (A \in \mathfrak{G}). \tag{1.5.2}$$

Denote with

$$\varphi_{|A} := \varphi(\cdot \cap A) \qquad \big(\varphi \in M(G), A \in \mathfrak{G}\big) \tag{1.5.3}$$

the subconfiguration of φ obtained by ignoring all points outside of A and identify $M(A)$ with the set of point configurations concentrated on A, i.e.

$$M(A) \cong \{\varphi \in M(G) : \varphi = \varphi_{|A}\} \qquad (A \in \mathfrak{G}). \tag{1.5.4}$$

With the aid of Lemma 1.5.2 we will then be prepared to find a product decomposition of Fock space.

Lemma 1.5.2 (see [19, Lemma 2.7]) *For disjoint $A_1, A_2 \in \mathfrak{G}$ the function $\varphi \mapsto (\varphi_{|A_1}, \varphi_{|A_2})$ is measurable and maps $M(A_1 \cup A_2)$ one-to-one onto $M(A_1) \times M(A_2)$. The image of $F_{\mu_{|A_1 \cup A_2}}$ by this map is equal to $F_{\mu_{|A_1}} \otimes F_{\mu_{|A_2}}$.*

Proof: That this map is bijective follows from the unique representation

$$\varphi = \varphi_{|A_1} + \varphi_{|A_2} \qquad \big(\varphi \in M(A_1 \cup A_2)\big).$$

It is measurable, because sets of the form

$$\{\varphi \in M(A_1) : \varphi(B_1) = k_1\} \times \{\varphi \in M(A_2) : \varphi(B_2) = k_2\}$$

for bounded $B_1 \in A_1 \cap \mathfrak{G}, B_2 \in A_2 \cap \mathfrak{G}$ and $k_1, k_2 \in \mathbb{N}_0$, generate $\mathfrak{M}(A_1) \otimes \mathfrak{M}(A_2)$ and their pre-image is equal to

$$\{\varphi \in M(A_1 \cup A_2) : \varphi(B_1) = k_1\} \cap \{\varphi \in M(A_1 \cup A_2) : \varphi(B_2) = k_2\},$$

which is a measurable subsets of $M(A_1 \cup A_2)$.

Denote the map in question with s. Using $d\underline{x}^n := d\mu^{\otimes n}(x_1, \ldots, x_n)$ and $d\underline{x}^0 := d\delta_{\mathcal{O}}$ for brevity, the proof will be completed with reference to transformation rule for integrals and the following chain of equations, which is true for all integrable (or non-negative) $h \in \mathbb{M}\big(M(A_1) \times M(A_2)\big)$.

$$\begin{aligned}
\int h \circ s \, dF_{\mu_{|A_1 \cup A_2}} &= \int h(\varphi_{|A_1}, \varphi_{|A_2}) \, dF_{\mu_{|A_1 \cup A_2}}(\varphi) \\
&= \sum_{n=0}^{\infty} \int_{M_n(A_1 \cup A_2)} h(\varphi_{|A_1}, \varphi_{|A_2}) \, dF_{\mu_{|A_1 \cup A_2}}(\varphi) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(A_1 \cup A_2)^n} h\left(\left(\sum_{i=1}^{n} \delta_{x_i}\right)_{|A_1}, \left(\sum_{i=1}^{n} \delta_{x_i}\right)_{|A_2}\right) d\underline{x}^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \int_{A_1^k} \int_{A_2^{n-k}} h\left(\sum_{i=1}^{k} \delta_{x_i}, \sum_{j=1}^{n-k} \delta_{y_j}\right) d\underline{y}^{n-k} \, d\underline{x}^k \\
&= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{1}{n!} \binom{n}{k} \int_{A_1^k} \int_{A_2^{n-k}} h\left(\sum_{i=1}^{k} \delta_{x_i}, \sum_{j=1}^{n-k} \delta_{y_j}\right) d\underline{y}^{n-k} \, d\underline{x}^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \int_{A_1^k} \sum_{m=0}^{\infty} \frac{1}{m!} \int_{A_2^m} h\left(\sum_{i=1}^{k} \delta_{x_i}, \sum_{j=1}^{m} \delta_{y_j}\right) d\underline{y}^{m} \, d\underline{x}^k \\
&= \sum_{k=0}^{\infty} \int_{M_k(A_1)} \sum_{m=0}^{\infty} \int_{M_m(A_2)} h\left(\varphi_1, \varphi_2\right) \, dF_{\mu_{|A_2}}(\varphi_2) \, dF_{\mu_{|A_1}}(\varphi_1) \\
&= \int h(\varphi_1, \varphi_2) \, dF_{\mu_{|A_1}}(\varphi_1) \, dF_{\mu_{|A_2}}(\varphi_2) = \int h \, d\left(F_{\mu_{|A_1}} \otimes F_{\mu_{|A_2}}\right).
\end{aligned}$$

■

For disjoint $A_1, A_2 \in \mathfrak{G}$ we will therefore identify

$$\big(M(A_1 \cup A_2), F_{\mu_{|A_1 \cup A_2}}\big) = \big(M(A_1) \times M(A_2), F_{\mu_{|A_1}} \otimes F_{\mu_{|A_2}}\big) \tag{1.5.5}$$

and recall the well-known isomorphism (see [61, Theorem II.10]) of

$$L^2\big(M(A_1) \times M(A_2)\big) \cong L^2\big(M(A_1)\big) \otimes L^2\big(M(A_2)\big), \tag{1.5.6}$$

leading to

$$\mathcal{M}(A_1 \cup A_2) \cong \mathcal{M}(A_1) \otimes \mathcal{M}(A_2) \tag{1.5.7}$$

with the property that

$$\Phi_1 \otimes \Phi_2(\varphi) = \Phi_1(\varphi_{|A_1}) \cdot \Phi_2(\varphi_{|A_2}) \qquad \big(\Phi_i \in \mathcal{M}(A_i), \varphi \in M(A_1 \cup A_2)\big). \tag{1.5.8}$$

We also observe, that, by induction, identification (1.5.5) easily extends to finitely many $A_1, \dots, A_n \in \mathfrak{G}$. Thus, we have a unique representation of

$$\varphi = \varphi_{|A_1} + \dots + \varphi_{|A_n} \qquad \big(\varphi \in M(A), (A_1, \dots, A_n) \in \Sigma(A), A \in \mathfrak{G}\big), \tag{1.5.9}$$

where $\Sigma(A)$ denotes the set of all finite measurable decompositions of $A \in \mathfrak{G}$ into disjoint $A_1, \ldots, A_n \in \mathfrak{G}$ $\big($see (1.5.1)$\big)$.

Moreover, definition of the coherent function $\big($see (1.4.1)$\big)$ and (1.5.9) imply

$$\exp[f]\,(\varphi) = \prod_{k=1}^{n} \exp\big[f_{|A_k}\big]\,(\varphi_{|A_k})$$
$$\big(f \in L^2(A), \varphi \in M(A), (A_1, \ldots, A_n) \in \Sigma(A), A \in \mathfrak{G}\big), \quad (1.5.10)$$

which, together with (1.5.7) and (1.5.8), leads to a natural identification of

$$\mathcal{M}(A) \cong \bigotimes_{k=1}^{n} \mathcal{M}(A_k) \qquad \big((A_1, \ldots, A_n) \in \Sigma(A), A \in \mathfrak{G}\big) \tag{1.5.11}$$

and

$$\exp[f] = \bigotimes_{k=1}^{n} \exp\big[f_{|A_k}\big] \qquad \big(f \in L^2(A), (A_1, \ldots, A_n) \in \Sigma(A), A \in \mathfrak{G}\big). \tag{1.5.12}$$

Remark 1.5.3 *Referring to notation of remark 1.3.4, it was shown in [60, Proposition 19.6] that for separable Hilbert spaces H_1, H_2*

$$\Gamma\big(H_1 \oplus H_2\big) \cong \Gamma(H_1) \otimes \Gamma(H_2) \tag{1.5.13}$$

under the isomorphism extending

$$\exp[f \oplus g] \mapsto \exp[f] \otimes \exp[g] \qquad \big(f \in H_1, g \in H_2\big). \tag{1.5.14}$$

Together with the fact that for $(A_1, \ldots, A_n) \in \Sigma(A), A \in \mathfrak{G}$

$$L^2(A) \cong L^2(A_1) \oplus \ldots \oplus L^2(A_n), \tag{1.5.15}$$

(1.5.11) *reads as*

$$\Gamma\big(L^2(A)\big) \cong \Gamma\big(L^2(A_1) \oplus \ldots \oplus L^2(A_n)\big) \cong \bigotimes_{k=1}^{n} \Gamma\big(L^2(A_k)\big). \tag{1.5.16}$$

Remark 1.5.4 *For $A \in \mathfrak{G}$ denote with I_A the isomorphism between $\Gamma\big(L^2(A)\big)$ and $\mathcal{M}(A)$. Then simple calculations with exponential vectors show that I_A is compatible with the factorisation property of Fock space, i.e.*

$$I_A = \bigotimes_{k=1}^{n} I_{A_k} \qquad \big((A_1, \ldots, A_n) \in \Sigma(A)\big).$$

The same is true for all the other isomorphic representations of Fock space considered in this and of multiple Fock space in the next section. Because of this compatibility, we will continue to omit the isomorphisms in questions in all future calculations.

For $A \in \mathfrak{G}$ and $A_1, \ldots, A_n \in \Sigma(A)$ we call the identifications developed in this section *regional decomposition* or *regional factorisation* and summarise them in the following table:

Representation	Compound	Decomposed
Configuration	$\varphi = \sum_{x \in \varphi} \delta_x$	$(\varphi_{\vert A_1}, \dots, \varphi_{\vert A_n})$ with $\varphi_{\vert A_k} = \sum_{x \in \varphi, x \in A_k} \delta_x$
Configuration Space	$\big(M(A), F_{\mu_{\vert A}}\big)$	$\left(\bigtimes_{k=1}^{n} M(A_k), \bigotimes_{k=1}^{n} F_{\mu_{\vert A_k}} \right)$
Fock Space	$L^2\big(M(A)\big)$ $= \mathcal{M}(A)$ $\cong \Gamma\big(L^2(A)\big)$ $\cong \Gamma\left(\bigoplus_{k=1}^{n} L^2(A_k) \right)$	$L^2\left(\bigtimes_{k=1}^{n} M(A_k) \right)$ $\cong \bigotimes_{k=1}^{n} \mathcal{M}(A_k)$ $\cong \bigotimes_{k=1}^{n} \Gamma\big(L^2(A_k)\big)$
Coherent Function	$\exp[f]\,(\varphi) = \prod_{x \in \varphi} f(x)$	$= \prod_{k=1}^{n} \exp\big[f_{\vert A_k}\big]\,(\varphi_{\vert A_k})$
Exponential Vector	$\exp[f] \cong \exp\left[\bigoplus_{k=1}^{n} f_{\vert A_k} \right]$	$\bigotimes_{k=1}^{n} \exp\big[f_{\vert A_k}\big]$

Table 1.1: Factorisation/Decomposition of Fock Space

1.6 Multiple Fock Space and its Factorisation by Parts and Regions

We now want to find a description of several quantum point systems as a single joint system. Usually this is done by taking the tensor product of the respective Fock spaces. Another approach would be to consider the Fock space on multiple point configurations (see [25, p.9]). During this section it will be shown that both descriptions are in fact isomorphic. Let $d \in \mathbb{N}$ be a fixed positive integer. Denote

$$\mu_{\vert A}^{(d)} := \mu_{\vert A} \otimes \sum_{i=1}^{d} \delta_i \qquad \text{and} \qquad A^{(d)} := A \times \{1, \dots, d\} \qquad (d \in \mathbb{N}, A \in \mathfrak{G}). \quad (1.6.1)$$

This means that each point $(x, i) \in A^{(d)}$ is not only characterised by its position $x \in A$ but also its "mark" $i \in \{1, \dots, d\}$, determining which of the d partial configurations it belongs to. We oberserve that $A^{(d)}$ is again a separable, metric space and $\mu_{\vert A}^{(d)}$ a locally finite, non-atomic measure and therefore, we may consider the symmetric Fock space

$$\mathcal{M}\big(A^{(d)}\big) = L^2\Big(M\big(A^{(d)}\big), F_{\mu^{(d)}_{|A}}\Big).$$

Since $\big(A \times \{1\}, \ldots, A \times \{d\}\big) \in \Sigma\big(A^{(d)}\big)$ and using (1.5.5), we may identify

$$\Big(M\big(A^{(d)}\big), F_{\mu^{(d)}_{|A}}\Big) = \big([M(A)]^d, F^{\otimes d}_{\mu_{|A}}\big) \tag{1.6.2}$$

via

$$\varphi = (\varphi_1, \ldots, \varphi_d) \qquad \Big(\varphi \in M\big(A^{(d)}\big)\Big), \tag{1.6.3}$$

where

$$\varphi_i := \varphi_{|A\times\{i\}} \in M\big(A \times \{i\}\big) \cong M(A) \qquad (1 \leq i \leq d). \tag{1.6.4}$$

Similarly,

$$f = (f_1, \ldots, f_d) \qquad \Big(f \in L^2\big(A^{(d)}\big)\Big), \tag{1.6.5}$$

where

$$f_i := f_{|A\times\{i\}} \in L^2\big(A \times \{i\}\big) \cong L^2(A) \qquad (1 \leq i \leq d), \tag{1.6.6}$$

and therefore, by (1.5.12),

$$\exp[f] = \bigotimes_{i=1}^{d} \exp[f_i] \qquad \Big(f \in L^2\big(A^{(d)}\big)\Big). \tag{1.6.7}$$

Using (1.5.11), this leads to another natural identification of

$$\mathcal{M}\big(A^{(d)}\big) \cong \bigotimes_{i=1}^{d} \mathcal{M}\big(A \times \{i\}\big) \cong \bigotimes_{i=1}^{d} \mathcal{M}(A) =: \mathcal{M}^{\otimes d}(A), \tag{1.6.8}$$

i.e. the Fock space over d-ple configurations is isomorphic to the tensor product of d identical Fock spaces of single configurations.

Remark 1.6.1 *Using* (1.5.13) *and the fact that the orthogonal sum of d identical copies of $L^2(A)$ is isomorphic to $L^2\big(A, \mathbb{C}^d\big)$,* (1.6.8) *may be written as*

$$\Gamma^{\otimes d}\big(L^2(A)\big) := \bigotimes_{i=1}^{d} \Gamma\big(L^2(A)\big) \cong \Gamma\Big(L^2\big(A, \mathbb{C}^d\big)\Big). \tag{1.6.9}$$

We call the identifications above *decomposition (or factorisation) according to parts.* Of course, multiple Fock space can also be decomposed according to regions. Thereby we introduce as a short-hand for $A \in \mathfrak{G}$:

$$\varphi_{|A} := \varphi_{|A^{(d)}} \in M\big(A^{(d)}\big) \qquad \Big(\varphi \in M\big(G^{(d)}\big)\Big), \tag{1.6.10}$$

$$f_{|A} := f_{|A^{(d)}} \in L^2\big(A^{(d)}\big) \qquad \Big(f \in L^2\big(G^{(d)}\big)\Big), \tag{1.6.11}$$

and

$$\psi_{|A} := \psi_{|M(A^{(d)})} \in \mathcal{M}\big(A^{(d)}\big) \qquad \Big(\psi \in \mathcal{M}\big(G^{(d)}\big)\Big). \tag{1.6.12}$$

This implies in particular

$$\exp[f]_{|A} = \exp[f]_{|M(A^{(d)})} = \exp\big[f_{|A}\big] \qquad \Big(f \in L^2\big(G^{(d)}\big), A \in \mathfrak{G}\Big). \tag{1.6.13}$$

We summarise decomposition of multiple Fock space according to parts and regions as follows:

Representation	Compound	Decomposition According to	
		Parts	Regions
Configuration	$\varphi = \sum_{(x,j)\in\varphi} \delta_{(x,j)}$	$(\varphi_1, \ldots, \varphi_d)$ with $\varphi_i = \sum_{(x,i)\in\varphi} \delta_x$	$(\varphi_{\vert A_k})_{k=1}^n$ with $\varphi_{\vert A} = \sum_{(x,j)\in\varphi, x\in A} \delta_{(x,j)}$
Config. Space	$\Big(M\big(A^{(d)}\big), F_{\mu^{(d)}_{\vert A}}\Big)$	$\big([M(A)]^d, F^{\otimes d}_{\mu_{\vert A}}\big)$	$\Big(\bigtimes_{k=1}^{n} M\big(A_k^{(d)}\big), \bigotimes_{k=1}^{n} F_{\mu^{(d)}_{\vert A_k}}\Big)$
Fock Space	$L^2\Big(M\big(A^{(d)}\big)\Big)$ $= \mathcal{M}\big(A^{(d)}\big)$ $\cong \Gamma\Big(L^2\big(A^{(d)}\big)\Big)$ $\cong \Gamma\big([L^2(A)]^{\oplus d}\big)$ $\cong \Gamma\big(L^2(A, \mathbb{C}^d)\big)$	$L^2\big([M(A)]^d\big)$ $\cong \mathcal{M}^{\otimes d}(A)$ $\cong \Gamma^{\otimes d}\big(L^2(A)\big)$	$L^2\left(\bigtimes_{k=1}^{n} M\big(A_k^{(d)}\big)\right)$ $\cong \bigotimes_{k=1}^{n} \mathcal{M}\big(A_k^{(d)}\big)$ $\cong \bigotimes_{k=1}^{n} \Gamma\Big(L^2\big(A_k^{(d)}\big)\Big)$ $\cong \bigotimes_{k=1}^{n} \Gamma\big([L^2(A_k)]^{\oplus d}\big)$ $\cong \bigotimes_{k=1}^{n} \Gamma\big(L^2(A_k, \mathbb{C}^d)\big)$
Coh. Function	$\exp[f]\,(\varphi)$ $= \prod_{(x,i)\in\varphi} f(x,i)$	$= \prod_{i=1}^{d} \exp[f_i]\,(\varphi_i)$ with $f_i = f(\cdot, i)$	$= \prod_{k=1}^{n} \exp\big[f_{\vert A_k}\big]\,(\varphi_{\vert A_k})$ with $f_{\vert A} = f_{\vert A^{(d)}}$
Exp. Vector	$\exp[f]$ $\cong \exp\left[\bigoplus_{i=1}^{d} f_i\right]$	$\bigotimes_{i=1}^{d} \exp[f_i]$	$\bigotimes_{k=1}^{n} \exp\big[f_{\vert A_k}\big]$

Table 1.2: Factorisation/Decomposition of Multiple Fock Space

2 Factorisation of Vectors and Operators

Having seen factorisability of (multiple) Fock space in sections 1.5 and 1.6, this chapter is dedicated to the characterisation of factorisable elements of and isometric operators on multiple Fock space. To this end we will make use of the decomposition of multiple Fock space according to regions as summarised in table 1.2. The main results are summarised in section 2.1 and developed in detail in sections 2.2 and 2.3. In an algebraic setting, they can also be found in [3] and [44, Theorems 5.1, 5.2]. Necessary auxiliary results are given in sections 2.4, 2.5 and 2.6.

Note also that, different from chapter 1, confinement to non-atomic primary measure μ will be vital now and in the sequel.

2.1 Summary

In [44], non-zero $\psi \in H$ or $B \in \mathcal{B}(H)$ for some $\mathfrak{G}$-factorisable H is called factorisable, if it factorises for all finite decompositions $G_1, \ldots, G_n$ of G. We only require factorisation according to A and A^c, but with the additional assumption that $\psi(\mathcal{O}) \neq 0$ or $B\exp[f]\,(\mathcal{O}) \neq 0$ for all $f \in \mathcal{M}\big(G^{(d)}\big)$, respectively. In $\mathcal{M}\big(G^{(d)}\big)$ this turns out to be equivalent.

Definition 2.1.1 *$\psi \in \mathcal{M}\big(G^{(d)}\big)$ with $\psi(\mathcal{O}) \neq 0$ is called A-factorisable for some $A \in \mathfrak{G}$, if there exist $\psi_A \in \mathcal{M}\big(A^{(d)}\big)$ and $\psi_{A^c} \in \mathcal{M}\big((A^c)^{(d)}\big)$ such that $\psi = \psi_A \otimes \psi_{A^c}$. This means that*

$$\psi(\varphi) = \psi_A(\varphi_{|A}) \cdot \psi_{A^c}(\varphi_{|A^c}) \qquad \Big(F_{\mu^{(d)}}\text{-a.e. } \varphi \in M\big(G^{(d)}\big)\Big). \tag{2.1.1}$$

ψ is called $\mathfrak{G}$-factorisable, if it is A-factorisable for every $A \in \mathfrak{G}$.

The set of $\mathfrak{G}$-factorisable functions in $\mathcal{M}\big(G^{(d)}\big)$ is given exactly by $\mathbb{C}\exp\big[L^2\big(G^{(d)}\big)\big]$. More precisely:

Theorem 2.1.2 (Summary of Theorem 2.2.6) *$\psi \in \mathcal{M}\big(G^{(d)}\big)$ is $\mathfrak{G}$-factorisable if and* *only if there exists unique* $f \in L^2\big(G^{(d)}\big)$ *such that*

$$\psi = \psi(\mathcal{O}) \cdot \exp[f]\,.$$

■

And what about factorisable operators?

Definition 2.1.3 *A bounded operator* $B : \mathcal{M}\big(G^{(d_1)}\big) \to \mathcal{M}\big(G^{(d_2)}\big)$ *such that* $(B\exp[f])(\mathcal{O}) \neq 0$ *for all* $f \in L^2\big(G^{(d_1)}\big)$ *is called A-factorisable for some* $A \in \mathfrak{G}$, *if* $B = B_A \otimes B_{A^c}$ *for some bounded operators* $B_A : \mathcal{M}\big(A^{(d_1)}\big) \to \mathcal{M}\big(A^{(d_2)}\big)$ *and* $B_{A^c} : \mathcal{M}\big((A^c)^{(d_1)}\big) \to \mathcal{M}\big((A^c)^{(d_2)}\big)$. *It is called* $\mathfrak{G}$*-factorisable, if it is A-factorisable for every* $A \in \mathfrak{G}$.

We have

Theorem 2.1.4 (Summary of Theorem 2.3.7) $\mathfrak{G}$*-factorisable isometries* $V : \mathcal{M}\big(G^{(d_1)}\big) \to \mathcal{M}\big(G^{(d_2)}\big)$ *are given by operators of the kind*

$$V = c \cdot \mathcal{W}(h)\Gamma(T) \tag{2.1.2}$$

for $c \in \mathbb{C}, |c| = 1, h \in L^2\big(G^{(d_2)}\big)$ *and isometric* $T : L^2\big(G^{(d_1)}\big) \to L^2\big(G^{(d_2)}\big)$. *In addition* T *preserves local subspaces in the sense of*

$$T(\chi_A g) = \chi_A T g \quad \Big(g \in L^2\big(G^{(d_1)}\big), A \in \mathfrak{G}\Big). \tag{2.1.3}$$

■

Thereby we used χ_A as a short-hand for $\chi_{A^{(d)}}$ in (2.1.3), which means that

$$\chi_A f = (\chi_A f_1, \ldots, \chi_A f_d) \qquad \Big(f \in L^2\big(G^{(d)}\big), A \in \mathfrak{G}, d \in \mathbb{N}\Big).$$

Now the question arises, what are the candidates for isometries T in theorem 2.1.4? Does (2.1.3) admit any further characterisation of T? Before answering this question in chapter 3, we will turn to giving the details on factorisable vectors and operators in the next sections.

2.2 $\mathfrak{G}$-Factorisable Functions

At the conclusion of this section (see Theorem 2.2.6) we will find that, up to a multiplicative constant, exponential vectors are the only $\mathfrak{G}$-factorisable functions from multiple Fock space $\mathcal{M}\big(G^{(d)}\big)$. In an algebraic setting $\mathfrak{G}$-factorisable functions were also characterised in [44, Theorem 5.1].

Throughout this section $d \in \mathbb{N}$ will be a fixed positive integer.

Recall (see definition 2.1.1) that $\psi \in \mathcal{M}\big(G^{(d)}\big)$ such that $\psi(\mathcal{O}) \neq 0$ is called A-factorisable for some $A \in \mathfrak{G}$, if $\psi = \psi_A \otimes \psi_{A^c}$, i.e.

$$\psi(\varphi) = \psi_A(\varphi_{|A}) \cdot \psi_{A^c}(\varphi_{|A^c}) \qquad \Big(F_{\mu^{(d)}}\text{-a.e. } \varphi \in M\big(G^{(d)}\big)\Big) \tag{2.2.1}$$

for some $\psi_A \in \mathcal{M}\big(A^{(d)}\big)$ and $\psi_{A^c} \in \mathcal{M}\big((A^c)^{(d)}\big)$. ψ is called $\mathfrak{G}$-factorisable, if it is A-factorisable for every $A \in \mathfrak{G}$.

Remark 2.2.1 *In [44], $\psi \in H$ for some $\mathfrak{G}$-factorisable H is called factorisable, if it factorises for all finite decompositions $G_1, \ldots, G_n$ of G, not only into A and A^c. But this turns out to be sufficient in our case (see Lemma 2.2.4).*

First we show, that for $\psi(\mathcal{O}) = 1$, factors are unique up to a multiplicative constant.

Lemma 2.2.2 *For some $A \in \mathfrak{G}$ let $\psi_A \in \mathcal{M}\big(A^{(d)}\big)$, $\psi_{A^c} \in \mathcal{M}\big((A^c)^{(d)}\big)$ and*

$$\psi := \psi_A \otimes \psi_{A^c} \in \mathcal{M}\big(G^{(d)}\big) \qquad \textit{such that} \qquad \psi(\mathcal{O}) = 1. \tag{2.2.2}$$

Then

$$\psi_A = \psi_A(\mathcal{O}) \cdot \psi_{|A}. \tag{2.2.3}$$

In particular $\psi = \psi_A \otimes \psi_{A^c} = \exp[f]$ for some $f \in L^2\big(G^{(d)}\big)$ implies

$$\psi_A = \psi_A(\mathcal{O}) \cdot \exp\big[f_{|A}\big]\,. \tag{2.2.4}$$

Proof: Fix $A \in \mathfrak{G}$, $\psi_A \in \mathcal{M}\big(A^{(d)}\big)$, $\psi_{A^c} \in \mathcal{M}\big((A^c)^{(d)}\big)$ such that (2.2.2) holds. Then

$$\psi_A(\mathcal{O}) \cdot \psi_{A^c}(\mathcal{O}) = \psi(\mathcal{O}) = 1$$

and

$$\varphi_{|A} = \varphi \quad \text{and} \quad \varphi_{|A^c} = \mathcal{O} \qquad \Big(\varphi \in M\big(A^{(d)}\big)\Big).$$

Thus we conclude from (2.2.2)

$$\psi_A(\mathcal{O}) \cdot \psi(\varphi) = \psi_A(\mathcal{O}) \cdot \psi_A(\varphi) \cdot \psi_{A^c}(\mathcal{O}) = \psi_A(\varphi) \qquad \Big(F_{\mu^{(d)}}\text{-a.e. } \varphi \in M\big(A^{(d)}\big)\Big). \tag{2.2.5}$$

The second part follows from

$$\psi_{|A} = \exp[f]_{|A} = \exp\bigl[f_{|A}\bigr] \qquad \Bigl(f \in L^2\bigl(G^{(d)}\bigr), A \in \mathfrak{G}\Bigr). \tag{2.2.6}$$

■

Lemma 2.2.2 implies, that if ψ such that $\psi(\mathcal{O}) = 1$ factorises for some ψ_A, ψ_{A^c}, then it factorises according to $\psi = \psi_{|A} \otimes \psi_{|A^c}$. We have

Corollary 2.2.3 *For some $A \in \mathfrak{G}$ let $\psi_A \in \mathcal{M}\bigl(A^{(d)}\bigr)$, $\psi_{A^c} \in \mathcal{M}\bigl((A^c)^{(d)}\bigr)$ and*

$$\psi := \psi_A \otimes \psi_{A^c} \in \mathcal{M}\bigl(G^{(d)}\bigr) \qquad \textit{such that} \qquad \psi(\mathcal{O}) = 1. \tag{2.2.7}$$

Then

$$\psi_{|A\cup B} = \psi_{|A} \otimes \psi_{|B} \qquad (B \in \mathfrak{G}, B \subseteq A^c). \tag{2.2.8}$$

Proof: Fix disjoint $A, B \in \mathfrak{G}$ and $\psi = \psi_A \otimes \psi_{A^c} \in \mathcal{M}\bigl(G^{(d)}\bigr)$ such that $\psi(\mathcal{O}) = 1$. Using Lemma 2.2.2, we have

$$\begin{aligned} \psi(\varphi) = \psi_A(\varphi_{|A}) \cdot \psi_{A^c}(\varphi_{|B}) &= \psi_A(\mathcal{O}) \cdot \psi(\varphi_{|A}) \cdot \psi_{A^c}(\mathcal{O}) \cdot \psi(\varphi_{|B}) \\ &= \psi(\mathcal{O}) \cdot \psi(\varphi_{|A}) \cdot \psi(\varphi_{|B}) = \psi(\varphi_{|A}) \cdot \psi(\varphi_{|B}) \\ &\qquad \Bigl(F_{\mu^{(d)}}\text{-a.e. } \varphi \in M\bigl((A \cup B)^{(d)}\bigr)\Bigr). \end{aligned} \tag{2.2.9}$$

■

Corollary 2.2.3 now shows, that if ψ such that $\psi(\mathcal{O}) = 1$ factorises for all $A \in \mathfrak{G}$ and some ψ_A, ψ_{A^c}, then it is "$\mathfrak{G}$-factorisable for all finite decompositions", not only according to A and A^c (see remark 2.2.1).

Lemma 2.2.4 *Assume that for all $A \in \mathfrak{G}$ there exist $\psi_A \in \mathcal{M}\bigl(A^{(d)}\bigr)$ and $\psi_{A^c} \in \mathcal{M}\bigl((A^c)^{(d)}\bigr)$ such that*

$$\psi = \psi_A \otimes \psi_{A^c} \in \mathcal{M}\bigl(G^{(d)}\bigr) \qquad \textit{and} \qquad \psi(\mathcal{O}) = 1. \tag{2.2.10}$$

Then

$$\psi = \bigotimes_{k=1}^{n} \psi_{|G_k} \qquad \bigl(n \in \mathbb{N}, (G_1, \ldots, G_n) \in \Sigma(G)\bigr). \tag{2.2.11}$$

Proof: Fix $\mathfrak{G}$-factorisable $\psi \in \mathcal{M}\bigl(G^{(d)}\bigr)$ such that $\psi(\mathcal{O}) = 1$. We will show (2.2.11) by induction. For $n = 1$ it is trivial. Now assume (2.2.11) for some $n \in \mathbb{N}$. Then, by the induction hypothesis and Corollary 2.2.3,

$$\psi = \bigotimes_{k=1}^{n-1} \psi_{|G_k} \otimes \psi_{|G_n \cup G_{n+1}} = \bigotimes_{k=1}^{n+1} \psi_{|G_k} \qquad \bigl((G_1, \ldots, G_{n+1}) \in \Sigma(G)\bigr). \tag{2.2.12}$$

■

We give a first characterisation of factorisable functions from $\mathcal{M}\bigl(G^{(d)}\bigr)$:

Proposition 2.2.5 *For $\psi \in \mathcal{M}\big(G^{(d)}\big)$ (I) and (II) are equivalent:*

(I) For all $A \in \mathfrak{G}$ and some $\psi_A \in \mathcal{M}\big(A^{(d)}\big)$ and $\psi_{A^c} \in \mathcal{M}\big((A^c)^{(d)}\big)$

$$\psi = \psi_A \otimes \psi_{A^c} \qquad \textit{and} \qquad \psi(\mathcal{O}) = 1. \tag{2.2.13}$$

(II) There exists a unique $f \in L^2\big(G^{(d)}\big)$ such that $\psi = \exp[f]$.

Proof: (II) implies (I): This is seen by virtue of

$$\psi = \exp[f] = \exp\big[f_{|A}\big] \otimes \exp\big[f_{|A^c}\big] \qquad (A \in \mathfrak{G}).$$

(I) implies (II): Fix $\psi \in \mathcal{M}\big(G^{(d)}\big)$ compliant with (I) and define

$$f(x,i) := \psi\left(\delta_{(x,i)}\right) \qquad \big((x,i) \in G^{(d)}\big). \tag{2.2.14}$$

Then (I) implies

$$\psi(\mathcal{O}) = 1 = \exp[f]\,(\mathcal{O}). \tag{2.2.15}$$

Now define

$$M_1^d(\sigma) := \Big\{\varphi \in M\big(G^{(d)}\big) : \varphi\big(G_k^{(d)}\big) = 1, k = 1, \ldots, n\Big\} \qquad \big(\sigma = (G_1, \ldots, G_n) \in \Sigma(G)\big) \tag{2.2.16}$$

to be the set of multiple point configurations such that its superposition has exactly one point in each G_i of the decomposition σ. Since there exists a sequence $(\sigma_m)_{m\in\mathbb{N}}$ in $\Sigma(G)$ such that

$$F_{\mu^{(d)}}\left(M\big(G^{(d)}\big) \setminus \bigcup_{m\in\mathbb{N}} M_1^d(\sigma_m)\right) = F_{\mu^{(d)}}(\{\mathcal{O}\}),$$

(see Corollary 2.4.4) and we already have (2.2.15) we are only left to show

$$\psi(\varphi) = \exp[f]\,(\varphi) \qquad \big(F_{\mu^{(d)}}\text{-a.e. } \varphi \in M_1^d(\sigma), \sigma \in \Sigma(G)\big). \tag{2.2.17}$$

Now fix $\sigma = (G_1, \ldots, G_n) \in \Sigma(G)$. Then $\varphi \in M_1^d(\sigma)$ has the form

$$\varphi = \sum_{k=1}^{n} \delta_{(x_k, i_k)}, \tag{2.2.18}$$

where $x_k \in G_k$ and $1 \leq i_k \leq d$ for $1 \leq k \leq n$. Lemma 2.2.4 and (2.2.14) therefore imply

$$\psi(\varphi) = \prod_{k=1}^{n} \psi(\varphi_{|G_k}) = \prod_{k=1}^{n} \psi(\delta_{(x_k,i_k)}) = \prod_{k=1}^{n} f(x_k, i_k) = \exp[f]\,(\varphi) \qquad \big(F_{\mu^{(d)}}\text{-a.e. } \varphi \in M_1^d(\sigma)\big). \tag{2.2.19}$$

But as σ was arbitrary, this shows (2.2.17).

Finally, uniqueness follows from Lemma 1.4.4 ($f \mapsto \exp[f]$ is one-to-one). ■

This is the final characterisation: The set of $\mathfrak{G}$-factorisable functions in $\mathcal{M}\big(G^{(d)}\big)$ is given exactly by $\mathbb{C}\exp\big[L^2\big(G^{(d)}\big)\big]$. More precisely:

Theorem 2.2.6 *For $\psi \in \mathcal{M}\big(G^{(d)}\big)$ (I) and (II) are equivalent:*

(I) ψ is $\mathfrak{G}$-factorisable.

(II) There exist unique $0 \neq c \in \mathbb{C}$ and $f \in L^2\big(G^{(d)}\big)$ such that $\psi = c \cdot \exp[f]$.

In this case $c = \psi(\mathcal{O})$ and $c \cdot f = \psi(\delta_{(\cdot,\cdot)})$.

Proof: Apply Proposition 2.2.5 to

$$\tilde{\psi} := \frac{\psi}{\psi(\mathcal{O})} \qquad \text{with} \qquad \tilde{\psi}_A = \frac{\psi_A}{\psi_A(\mathcal{O})} \qquad \text{and} \qquad \tilde{\psi}_{A^c} = \frac{\psi_{A^c}}{\psi_{A^c}(\mathcal{O})}.$$

■

2.3 $\mathfrak{G}$-Factorisable Operators

Throughout this section $d_1, d_2 \in \mathbb{N}$ will be arbitrary but fixed positive integers. In an algebraic setting $\mathfrak{G}$-factorisable unitaries were also characterised in [44, Theorem 5.2].

Recall (see definition 2.1.3) that bounded $B : \mathcal{M}\big(G^{(d_1)}\big) \to \mathcal{M}\big(G^{(d_2)}\big)$ such that $(B\exp[f])(\mathcal{O}) \neq 0$ for all $f \in L^2\big(G^{(d_1)}\big)$ is called A-factorisable for some $A \in \mathfrak{G}$, if $B = B_A \otimes B_{A^c}$ for some bounded operators $B_A : \mathcal{M}\big(A^{(d_1)}\big) \to \mathcal{M}\big(A^{(d_2)}\big)$ and $B_{A^c} : \mathcal{M}\big((A^c)^{(d_1)}\big) \to \mathcal{M}\big((A^c)^{(d_2)}\big)$. B is called $\mathfrak{G}$-factorisable, if it is A-factorisable for every $A \in \mathfrak{G}$.

Definition 2.3.1 *An operator $B : \mathcal{M}\big(A^{(d_1)}\big) \to \mathcal{M}\big(A^{(d_2)}\big)$ for some $A \in \mathfrak{G}$ is called vacuum-preserving, if $B\exp[0] = \exp[0]$.*

We also define $B_{|A} : \mathcal{M}\big(A^{(d_1)}\big) \to \mathcal{M}\big(A^{(d_2)}\big)$ by

$$B_{|A}\psi := \Big(B\big(\psi \otimes \exp\big[0_{|A^c}\big]\big)\Big)_{|A} \qquad \Big(\psi \in \mathcal{M}\big(A^{(d_1)}\big)\Big) \tag{2.3.1}$$

to be the (local) restriction of bounded $B : \mathcal{M}\big(G^{(d_1)}\big) \to \mathcal{M}\big(G^{(d_2)}\big)$ to $\mathcal{M}\big(A^{(d_1)}\big)$, $A \in \mathfrak{G}$. Observe that $B_{|A}$ is again linear and bounded with $\left\|B_{|A}\right\| \leq \|B\|$. For exponential vectors (2.3.1) reads as

$$B_{|A} \exp\big[f_{|A}\big] = \big(B \exp[\chi_A f]\big)_{|A} \qquad \Big(f \in L^2\big(G^{(d_1)}\big)\Big). \tag{2.3.2}$$

Then vacuum-preserving, $\mathfrak{G}$-factorisable operators can be characterised:

Proposition 2.3.2 *Let $B : \mathcal{M}\big(G^{(d_1)}\big) \to \mathcal{M}\big(G^{(d_2)}\big)$ be a vacuum-preserving, $\mathfrak{G}$-factorisable operator with the property that*

$$B \exp[g]\,(\mathcal{O}) = 1 \qquad \Big(g \in L^2\big(G^{(d_1)}\big)\Big). \tag{2.3.3}$$

Then there exists a unique everywhere-defined linear operator $T : L^2\big(G^{(d_1)}\big) \to L^2\big(G^{(d_2)}\big)$ such that

$$B \exp[g] = \exp[T(g)] \quad \text{and} \quad T(\chi_A h) = \chi_A T(h) \qquad \Big(g, h \in L^2\big(G^{(d_1)}\big), A \in \mathfrak{G}\Big). \tag{2.3.4}$$

In addition, $B = B_{|A} \otimes B_{|A^c}$ for all $A \in \mathfrak{G}$.

Proof: For all $A \in \mathfrak{G}$ fix B_A and B_{A^c} such that $B = B_A \otimes B_{A^c}$. Defining

$$\psi^g = B \exp[g] \qquad \text{and} \qquad \psi^g_A := B_A \exp\big[g_{|A}\big] \qquad \Big(g \in L^2\big(G^{(d_1)}\big), A \in \mathfrak{G}\Big), \tag{2.3.5}$$

(2.3.3) tells us that

$$\psi^g(\mathcal{O}) = 1 \qquad \Big(g \in L^2\big(G^{(d_1)}\big)\Big). \tag{2.3.6}$$

Moreover, because of $B = B_A \otimes B_{A^c}$ and (2.3.5),

$$\psi^g = \psi^g_A \otimes \psi^g_{A^c} \qquad \Big(g \in L^2\big(G^{(d_1)}\big), A \in \mathfrak{G}\Big). \tag{2.3.7}$$

Hence, by proposition 2.2.5, there exist unique $T(g) = \psi^g(\delta_{(\cdot)}) \in L^2\big(G^{(d_2)}\big)$ such that

$$\psi^g = \exp[T(g)] \qquad \Big(g \in L^2\big(G^{(d_1)}\big)\Big). \tag{2.3.8}$$

This defines the map $T : L^2\big(G^{(d_1)}\big) \to L^2\big(G^{(d_2)}\big)$ for which $T(0) = 0$, because B is vacuum-preserving. Now (2.3.5) implies

$$\psi^{\chi_A h}_A = \psi^h_A \quad \text{and} \quad \psi^{\chi_{A^c} h}_A = \psi^0_A \qquad \Big(h \in L^2\big(G^{(d_1)}\big), A \in \mathfrak{G}\Big). \tag{2.3.9}$$

Hence, for arbitrary $h \in L^2\big(G^{(d_1)}\big)$ and $A \in \mathfrak{G}$

$$\begin{aligned}
\exp[T(\chi_A h)] &= \psi^{\chi_A h} && \text{by (2.3.8)}\\
&= \psi_A^{\chi_A h} \otimes \psi_{A^c}^{\chi_A h} && \text{by (2.3.7)}\\
&= \psi_A^{h} \otimes \psi_{A^c}^{0} && \text{by (2.3.9)}\\
&= \big(\psi_A^{h}(\mathcal{O}) \cdot \psi_{|A}^{h}\big) \otimes \big(\psi_{A^c}^{0}(\mathcal{O}) \cdot \psi_{|A^c}^{0}\big) && \text{by Lemma 2.2.2}\\
&= \Big(\psi_A^{\chi_A h}(\mathcal{O}) \cdot \psi_{|A}^{h}\Big) \otimes \Big(\psi_{A^c}^{\chi_A h}(\mathcal{O}) \cdot \psi_{|A^c}^{0}\Big) && \text{by (2.3.9)}\\
&= \psi^{\chi_A h}(\mathcal{O}) \cdot \psi_{|A}^{h} \otimes \psi_{|A^c}^{0} && \text{by (2.3.7)}\\
&= \psi_{|A}^{h} \otimes \psi_{|A^c}^{0} && \text{by (2.3.6)}\\
&= \exp\big[T(h)_{|A}\big] \otimes \exp\big[T(0)_{|A^c}\big] && \text{by (2.3.8)}\\
&= \exp[\chi_A T(h)] && \text{by } T(0) = 0.
\end{aligned}$$

But the map $f \mapsto \exp[f]$ is one-to-one, thus showing (2.3.4). Now B and T fulfill the assumptions of Proposition 2.6.6, which implies that $B = B_{|A} \otimes B_{|A^c}$ and T is linear. ■

Remark 2.3.3 *The operator T from* (2.3.4) *acts locally in the sense that it preserves regional subspaces given by $A \in \mathfrak{G}$. Now the question arises, what kind of operators have this property? Is there any characterisation of local operators $T : L^2\big(G^{(d_1)}\big) \to L^2\big(G^{(d_2)}\big)$? The answer to this question will be given in chapter 3.*

Remark 2.3.4 *Lemmata 2.3.5 and 2.3.6 show, that the somewhat technical condition* (2.3.3) *in proposition 2.3.2, i.e. $B \exp[g]\,(\mathcal{O}) = 1$ for all $g \in L^2\big(G^{(d_1)}\big)$, is automatically fullfilled in case the vacuum-preserving operator B is isometric or such that B^* is also vacuum-preserving. For isometric B proposition 2.6.3 yields that T is also isometric.*

Lemma 2.3.5 *For $A \in \mathfrak{G}$ and isometric $V : \mathcal{M}\big(A^{(d_1)}\big) \to \mathcal{M}\big(A^{(d_2)}\big)$ the following four conditions are equivalent:*

(1) V is vacuum-preserving, i.e. $V \exp[0] = \exp[0]$.

(2) $(V\psi)(\mathcal{O}) = \psi(\mathcal{O}) \qquad \Big(\psi \in \mathcal{M}\big(A^{(d_1)}\big)\Big).$

(3) $(V \exp[f])(\mathcal{O}) = 1 \qquad \Big(f \in L^2\big(A^{(d_1)}\big)\Big).$

(4) $(V \exp[0])(\mathcal{O}) = 1.$

Proof: For $d \in \mathbb{N}$ the norm and scalar product in $\mathcal{M}\big(A^{(d)}\big)$ will be indexed by d.

(1) implies (2): We have

$$\psi(\mathcal{O}) = \langle \exp[0]\,, \psi\rangle_d \qquad \Big(\psi \in \mathcal{M}\big(A^{(d)}\big), d \in \mathbb{N}\Big) \tag{2.3.10}$$

and therefore, using (1) and the fact that V is isometric,

$$\begin{aligned}(V\psi)(\mathcal{O}) = \langle \exp[0]\,, V\psi\rangle_{d_2} &= \langle V\exp[0]\,, V\psi\rangle_{d_2} \\ &= \langle \exp[0]\,, \psi\rangle_{d_1} = \psi(\mathcal{O}) \qquad \Big(\psi \in \mathcal{M}\big(A^{(d_1)}\big)\Big). \end{aligned} \tag{2.3.11}$$

(2) implies (3) implies (4): Take $\psi = \exp[f]$ and $f = 0$, respectively.

(4) implies (1): We have

$$\|V\exp[0]\|_{d_2}^2 = \Big\|(V\exp[0])\chi_{M_0(A^{(d_2)})}\Big\|_{d_2}^2 + \Big\|(V\exp[0])\chi_{M_{>0}(A^{(d_2)})}\Big\|_{d_2}^2. \tag{2.3.12}$$

But using (4) and the fact that V is isometric

$$\Big\|(V\exp[0])\chi_{M_0(A^{(d_2)})}\Big\|_{d_2}^2 = |V\exp[0]\,(\mathcal{O})|^2 = 1 = \|\exp[0]\|_{d_1}^2 = \|V\exp[0]\|_{d_2}^2. \tag{2.3.13}$$

Hence $\Big\|(V\exp[0])\chi_{M_{>0}(A^{(d_2)})}\Big\|_{d_2}^2 = 0$. This and (4) shows (1). ■

For (only) bounded B we still have an implication similar to the first part of Lemma 2.3.5:

Lemma 2.3.6 *For $A \in \mathfrak{G}$ and bounded vacuum-preserving $B : \mathcal{M}\big(A^{(d_1)}\big) \to \mathcal{M}\big(A^{(d_2)}\big)$:*

$$(B^*\psi)(\mathcal{O}) = \psi(\mathcal{O}) \qquad \Big(\psi \in \mathcal{M}\big(A^{(d_2)}\big)\Big).$$

Proof: As in the proof of Lemma 2.3.5 we have for arbitrary $\psi \in \mathcal{M}\big(A^{(d_2)}\big)$

$$(B^*\psi)(\mathcal{O}) = \langle \exp[0]\,, B^*\psi\rangle_{d_1} = \langle B\exp[0]\,, \psi\rangle_{d_2} = \langle \exp[0]\,, \psi\rangle_{d_2} = \psi(\mathcal{O}).$$

■

Theorem 2.3.7 *$\mathfrak{G}$-factorisable isometries $V : \mathcal{M}\big(G^{(d_1)}\big) \to \mathcal{M}\big(G^{(d_2)}\big)$ are given by operators of the kind*

$$V = c \cdot \mathcal{W}(h)\Gamma(T) \tag{2.3.14}$$

for $c \in \mathbb{C}, |c| = 1, h \in L^2\big(G^{(d_2)}\big)$ and isometric $T : L^2\big(G^{(d_1)}\big) \to L^2\big(G^{(d_2)}\big)$. In addition T preserves local subspaces in the sense of

$$T(\chi_A g) = \chi_A T g \quad \Big(g \in L^2\big(G^{(d_1)}\big), A \in \mathfrak{G}\Big). \tag{2.3.15}$$

Proof: Setting $\psi_A^g := V_A \exp\left[g_{|A}\right]$ as in the proof of proposition 2.3.2 and using $\psi^g(\mathcal{O}) \neq 0$ (V is $\mathfrak{G}$-factorisable) shows that ψ^g is $\mathfrak{G}$-factorisable for all $g \in L^2\big(G^{(d_1)}\big)$. By proposition 2.2.5, V maps exponential vectors to multiples of exponential vectors and hence, by proposition 2.6.4,

$$V = c \cdot \mathcal{W}(h)\Gamma(T)$$

for $c \in \mathbb{C}, |c| = 1, h \in L^2\big(G^{(d_2)}\big)$ and isometric $T : L^2\big(G^{(d_1)}\big) \to L^2\big(G^{(d_2)}\big)$. We are left to show (2.3.15). But because of

$$\mathcal{W}(f) = \mathcal{W}\left(f_{|A}\right) \otimes \mathcal{W}\left(f_{|A^c}\right) \qquad \Big(f \in L^2\big(G^{(d_1)}\big), A \in \mathfrak{G}\Big)$$

and

$$\Gamma(T)\exp[f](\mathcal{O}) = \exp[Tf](\mathcal{O}) = 1 \qquad \Big(f \in L^2\big(G^{(d_1)}\big)\Big),$$

we have

$$\Gamma(T) = \frac{1}{c} \cdot \mathcal{W}(-h)\, V = \left(\frac{1}{c} \cdot \mathcal{W}\left(-h_{|A}\right) V_A\right) \otimes \Big(\mathcal{W}\left(-h_{|A^c}\right) V_{A^c}\Big)$$

for all $A \in \mathfrak{G}$, i.e. $\Gamma(T)$ is a vacuum-preserving, $\mathfrak{G}$-factorisable isometry. (2.3.15) therefore follows from application of proposition 2.3.2 and remark 2.3.4 to $\Gamma(T)$. $\blacksquare$

2.4 $\not\Sigma$-Lemma and Implications

We will first present the so-called $\not\Sigma$-Lemma (see [20], [36], [54] and [58]) which states, that, on Fock space, integrals with respect to $F_\mu^{\otimes d}$ on multiple configurations $(\varphi_1, \ldots, \varphi_d) \in [M(G)]^d$ may be reduced to integration with respect to F_μ on its superposition $\varphi = \varphi_1 + \cdots + \varphi_d \in M(G)$.

Using generalised binomial coefficients, a variant of the $\not\Sigma$-Lemma was also shown to hold for general (not necessarily non-atomic) underlying measures μ in [51] and with a rigorous proof in [63].

Lemma 2.4.1 ($\not\Sigma$-Lemma) *For $d \in \mathbb{N}$ let $h : [M(G)]^d \to \mathbb{C}$ be a measurable map, which is integrable with respect to $F_\mu^{\otimes d}$ or non-negative. Then*

$$\int\limits_{[M(G)]^d} h(\varphi_1, \ldots, \varphi_d)\, dF_\mu^{\otimes d}(\varphi_1, \ldots, \varphi_d) \quad = \quad \int\limits_{M(G)} \sum_{\varphi_1 + \cdots + \varphi_d = \varphi} h(\varphi_1, \ldots, \varphi_d)\, dF_\mu(\varphi).$$

$\blacksquare$

Since F_μ is concentrated on the set of finite and simple configurations $M^s_{<\infty}(G)$ (see Remark 1.3.2) and using identification

$$\Big(M\big(G^{(d)}\big), F_{\mu^{(d)}}\Big) = \big([M(G)]^d, F_\mu^{\otimes d}\big),$$

an immediate consequence of Lemma 2.4.1 is the fact, that $F_{\mu^{(d)}}$ is concentrated on

$$M^{s^*}_{<\infty}(G^{(d)}) := \left\{ \varphi \in M(G^{(d)}) : \varphi\big((\cdot)^{(d)}\big) = \sum_{(x,i)\in\varphi} \delta_x \in M^s_{<\infty}(G) \right\}, \tag{2.4.1}$$

the set of multiple configurations such that their superposition is simple and finite. In other words we have

Corollary 2.4.2

$$F_{\mu^{(d)}}\Big(M\big(G^{(d)}\big) \setminus M^{s^*}_{<\infty}\big(G^{(d)}\big)\Big) = 0.$$

■

Now define

$$M^d_1(\sigma) := \Big\{\varphi \in M\big(G^{(d)}\big) : \varphi\big(G_k^{(d)}\big) = 1, k = 1, \ldots, n\Big\}$$
$$\big(\sigma = (G_1, \ldots, G_n) \in \Sigma(G)\big). \tag{2.4.2}$$

to be the set of multiple point configurations such that their superposition has exactly one point in each G_i of the decomposition σ.

Lemma 2.4.3 *There exists a sequences $(\sigma_m)_{m\in\mathbb{N}}$ in $\Sigma(G)$, such that*

$$\bigcup_{m\in\mathbb{N}} M^d_1(\sigma_m) = M^{s^*}_{<\infty}\big(G^{(d)}\big) \setminus \{\mathcal{O}\}. \tag{2.4.3}$$

Proof: (I) " $\subseteq$ " : This is clear from definitions of $M^d_1(\sigma_m)$ $\big($see (2.4.2)$\big)$ and $M^{s^*}_{<\infty}\big(G^{(d)}\big)$ $\big($see (2.4.1)$\big)$.

" $\supseteq$ " : Denote with

$$B_\varepsilon(x) := \{y \in G : d(x, y) < \varepsilon\} \qquad (x \in G, \varepsilon > 0)$$

the open ball around x with radius ε. Also set

$$\delta(\varphi) := \min\{d(x, y) : (x, i), (y, j) \in \varphi, x \neq y\} \qquad \Big(\mathcal{O} \neq \varphi \in M^{s^*}_{<\infty}\big(G^{(d)}\big)\Big)$$

(with the convention that $\min \emptyset = \infty$) to be the distance of the two closest points in the superposition of φ.

Fixing a dense and countable set $D \subseteq G$ (G is separable) and denoting

$$\sigma_i(\varphi) := \begin{cases} G & n = 1 \\ \Big(B_{\frac{1}{i}}(x_1), \ldots, B_{\frac{1}{i}}(x_{n-1}), G \setminus \bigcup_{k=1}^{n-1} B_{\frac{1}{i}}(x_k)\Big) & n > 1 \end{cases}$$

for $i \in \mathbb{N}$ and $\varphi = \sum_{k=1}^{n} \delta_{(x_k, j_k)} \in M^{s^*}_{<\infty}\big(D^{(d)}\big)$ we observe that $\sigma_i(\varphi) \in \Sigma(G)$ whenever $i > \frac{2}{\delta(\varphi)}$. Now for every $\mathcal{O} \neq \varphi \in M^{s^*}_{<\infty}\big(G^{(d)}\big)$ and $i > \frac{4}{\delta(\varphi)}$ there exists $\varphi' \in M^{s^*}_{<\infty}\big(D^{(d)}\big)$ such that $\varphi \in M^d_1\big(\sigma_i(\varphi')\big)$. Hence

$$\bigcup_{\substack{i \in \mathbb{N} \\ \varphi' \in M^{s^*}_{<\infty}(D^{(d)})}} M^d_1\big(\sigma_i(\varphi')\big) \supseteq M^{s^*}_{<\infty}\big(G^{(d)}\big) \setminus \{\mathcal{O}\}.$$

The proof is complete, because the union on the left-hand side is a countable one. ■

Corollary 2.4.4 *There exists a sequence $(\sigma_m)_{m \in \mathbb{N}}$ in $\Sigma(G)$, such that*

$$F_{\mu^{(d)}}\left(M\big(G^{(d)}\big) \setminus \bigcup_{m \in \mathbb{N}} M^d_1(\sigma_m)\right) = F_{\mu^{(d)}}(\{\mathcal{O}\})$$

Proof: Lemma 2.4.3 and Corollary 2.4.2. ■

Remark 2.4.5 *For $d = 2$ this result was used but not proven in [32]. See also [56].*

Lemma 2.4.6 *There exists a sequence $(A_m)_{m \in \mathbb{N}}$ in $\mathfrak{G}$, such that $\mu(A^c_m) > 0$ for all $m \in \mathbb{N}$ and*

$$\bigcup_{m \in \mathbb{N}} M\big(A^{(d)}_m\big) = M_{<\infty}\big(G^{(d)}\big). \tag{2.4.4}$$

Proof: " $\subseteq$ " : Clear.

" $\supseteq$ " : Again, fixing a dense and countable set $D \subseteq G$, denote

$$A_i(\varphi) := \begin{cases} \emptyset & n = 0 \\ \bigcup_{k=1}^{n} B_{\frac{1}{i}}(x_k) & n > 0 \end{cases}$$

for $i \in \mathbb{N}$ and $\varphi = \sum_{k=1}^{n} \delta_{(x_k, j_k)} \in M_{<\infty}\big(D^{(d)}\big)$. We observe that for every $\varphi \in M_{<\infty}\big(G^{(d)}\big)$ there exist $\varphi' \in M_{<\infty}\big(D^{(d)}\big)$ and $i \in \mathbb{N}$ such that $\varphi \in M\big([A_i(\varphi')]^{(d)}\big)$ and $\mu\big(A_i^c(\varphi')\big) > 0$. Hence

$$\bigcup_{\substack{i \in \mathbb{N} \\ \varphi' \in M_{<\infty}(D^{(d)})}} M\big([A_i(\varphi')]^{(d)}\big) \supseteq M_{<\infty}\big(G^{(d)}\big),$$

thus completing the proof, because the union on the left-hand side is countable. ∎

Corollary 2.4.7 *There exists a sequence $(A_m)_{m \in \mathbb{N}}$ in $\mathfrak{G}$, such that $\mu(A_m^c) > 0$ for all $m \in \mathbb{N}$ and*

$$F_{\mu^{(d)}}\left(M\big(G^{(d)}\big) \setminus \bigcup_{m \in \mathbb{N}} M\big(A_m^{(d)}\big) \right) = 0. \tag{2.4.5}$$

Proof: $F_{\mu^{(d)}}$ is concentrated on $M_{<\infty}\big(G^{(d)}\big)$ and Lemma 2.4.6. ∎

2.5 Approximating exponential vectors

We will approximate exponential vectors of the kind $\exp[zf + zg]$ for $z \in \mathbb{C}$ and $f, g \in L^2\big(G^{(d)}\big)$ ($d \in \mathbb{N}$ is arbitrary but fixed throughout this section). It is a variation of the approximation by so-called *toy exponentials or toy Fock space* (see [57]).

Lemma 2.5.1 *Let $\big((a_{k,n})_{k=1}^{n}\big)_{n \in \mathbb{N}}$ be such that $a_{k,n} \geq 0 \quad (1 \leq k \leq n \in \mathbb{N})$,*

$$(I) \quad \sum_{k=1}^{n} a_{k,n} \underset{n \to \infty}{\longrightarrow} a \quad \textit{and} \qquad (II) \quad \alpha_n := \sup_{1 \leq k \leq n} a_{k,n} \underset{n \to \infty}{\longrightarrow} 0.$$

Then

$$\prod_{k=1}^{n} (1 + a_{k,n}) \underset{n \to \infty}{\longrightarrow} e^a.$$

Proof: Consider the function $f : [0, \infty) \to [0, \infty)$ defined through

$$f(x) := \begin{cases} \frac{\ln(1+x)}{x} & x > 0 \\ 1 & x = 0. \end{cases} \tag{2.5.1}$$

Then (a) f is monotonically decreasing, (b) $f(x) \underset{x \to 0}{\longrightarrow} 1$ and (c) $f \leq 1$.

Now fix $\big((a_{k,n})_{k=1}^{n}\big)_{n\in\mathbb{N}}$ for which the assumptions hold. From (2.5.1), (c) and (I) we conclude

$$\sum_{k=1}^{n}\ln(1+a_{k,n}) = \sum_{k=1}^{n} a_{k,n}f(a_{k,n}) \leq \sum_{k=1}^{n} a_{k,n} \qquad (n\in\mathbb{N}). \tag{2.5.2}$$

On the other hand, using (2.5.1), (II), (a) and (I), we have

$$\sum_{k=1}^{n}\ln(1+a_{k,n}) = \sum_{k=1}^{n} a_{k,n}f(a_{k,n}) \geq \sum_{k=1}^{n} a_{k,n}f(\alpha_n) \qquad (n\in\mathbb{N}). \tag{2.5.3}$$

(2.5.2) and (2.5.3) imply

$$\sum_{k=1}^{n} a_{k,n} \geq \sum_{k=1}^{n}\ln(1+a_{k,n}) \geq f(\alpha_n)\cdot\sum_{k=1}^{n} a_{k,n} \qquad (n\in\mathbb{N}).$$

Together with (I), (II) and (b) we have therefore shown that

$$\sum_{k=1}^{n}\ln(1+a_{k,n}) \underset{n\to\infty}{\longrightarrow} a,$$

which is equivalent to the assertion. ■

For a measurable decomposition $(G_1,\ldots,G_n)\in\Sigma(G)$ we identify

$$\mathcal{M}\big(G^{(d)}\big) \cong \bigotimes_{k=1}^{n}\mathcal{M}\big(G_k^{(d)}\big) \cong \bigotimes_{k,i=1}^{n,d}\mathcal{M}(G_k) \cong \bigotimes_{i=1}^{d}\mathcal{M}(G) \tag{2.5.4}$$

and for $A\in\mathfrak{G}$ denote with

$$M_{\leq 1}^{d}(A_1,\ldots,A_n) := \Big\{\varphi\in M\big(A^{(d)}\big) : \varphi\big(A_k^{(d)}\big)\leq 1, 1\leq k\leq n\Big\} \qquad \big((A_1,\ldots,A_n)\in\Sigma(A)\big) \tag{2.5.5}$$

the set of d-ple configurations in A having at most one point in each $A_k, 1\leq k\leq n$.

From (2.5.5) we immediately conclude

$$\chi_{M_{\leq 1}^{d}(G_1,\ldots,G_n)} = \bigotimes_{k=1}^{n}\chi_{M_{\leq 1}(G_k^{(d)})} \qquad \big((G_1,\ldots,G_n)\in\Sigma(G)\big) \tag{2.5.6}$$

and

$$\chi_{M_{\leq 1}(A^{(d)})} = \chi_{M_0(A^{(d)})} + \sum_{i=1}^{d}\big(\chi_{M_0(A\times\{1,\ldots,i-1\})}\otimes\chi_{M_0(A\times\{i\})}\otimes\chi_{M_0(A\times\{i+1,\ldots,d\})}\big) \qquad (A\in\mathfrak{G}), \tag{2.5.7}$$

where a factor $\chi_{M_0(A\times\emptyset)}$ appearing in the tensor product, which is the case for $i=1$ and $i=d$, is meant to be omitted.

Obviously, we also have

$$\chi_{M_0(A^{(d)})} = \bigotimes_{i=1}^{d} \chi_{M_0(A)} \qquad (A \in \mathfrak{G}). \tag{2.5.8}$$

For any measurable set Y we will write $\chi_Y^c := 1 - \chi_Y$, i.e. χ_Y^c is the indicator function of the complement of Y.

Lemma 2.5.2 *If $g \in L^2\big(G^{(d)}\big)$ and $\sigma^{(n)} := (G_{1,n}, \ldots, G_{n,n}) \in \Sigma(G), n \in \mathbb{N}$, such that*

$$(I) \qquad \sup_{1 \le k \le n} \big\| g_{|G_{k,n}} \big\| \underset{n\to\infty}{\longrightarrow} 0,$$

then

$$\left\| \exp[g] \cdot \chi^c_{M^d_{\le 1}\left(\sigma^{(n)}\right)} \right\| \underset{n\to\infty}{\longrightarrow} 0.$$

Proof: Fix $\left(\sigma^{(n)}\right)_{n\in\mathbb{N}}$ and $g = (g_1, \ldots, g_d) \in L^2\big(G^{(d)}\big)$ for which (I) holds.

The fact that $\exp[g] = \bigotimes_{k=1}^{n} \exp\big[g_{|G_{k,n}}\big]$ and (2.5.6) imply

$$\exp[g] \cdot \chi_{M^d_{\le 1}(\sigma^{(n)})} = \bigotimes_{k=1}^{n} \Big(\exp\big[g_{|G_{k,n}}\big] \cdot \chi_{M_{\le 1}(G^{(d)}_{k,n})} \Big) \qquad (n \in \mathbb{N}),$$

and therefore we have

$$\left\| \exp[g] \cdot \chi_{M^d_{\le 1}\left(\sigma^{(n)}\right)} \right\|^2 = \prod_{k=1}^{n} \left\| \exp\big[g_{|G_{k,n}}\big] \cdot \chi_{M_{\le 1}(G^{(d)}_{k,n})} \right\|^2 \qquad (n \in \mathbb{N}). \tag{2.5.9}$$

From (1.4.8) we get

$$\big\| \exp[f] \cdot \chi_{M_m(A)} \big\|^2 = \frac{\|f\|^{2m}}{m!} \qquad \big(m \in \mathbb{N}, A \in \mathfrak{G}, f \in L^2(A)\big). \tag{2.5.10}$$

Therefore,

$$\exp\big[g_{|A}\big] = \bigotimes_{i=1}^{d} \exp\big[g_{i|A}\big] \qquad (A \in \mathfrak{G}), \tag{2.5.11}$$

(2.5.8) and (2.5.10) imply

$$\big\| \exp\big[g_{|A}\big] \cdot \chi_{M_0(A^{(d)})} \big\|^2 = 1 \qquad (A \in \mathfrak{G}) \tag{2.5.12}$$

and, remembering the remark after (2.5.7),

$$\big\| \exp\big[g_{|A}\big] \cdot \chi_{M_0(A\times\{1,\ldots,i-1\})} \otimes \chi_{M_0(A\times\{i\})} \otimes \chi_{M_0(A\times\{i+1,\ldots,d\})} \big\|^2 = \big\| g_{i|A} \big\|^2$$
$$(A \in \mathfrak{G}, 1 \le i \le d). \tag{2.5.13}$$

Thus (2.5.7), (2.5.12), (2.5.13) and the fact that $\left\|g_{|A}\right\|^2 = \sum_{i=1}^{d} \left\|g_{i|A}\right\|^2$ show that

$$\left\|\exp\left[g_{|A}\right] \cdot \chi_{M_{\leq 1}(A^{(d)})}\right\|^2 = 1 + \left\|g_{|A}\right\|^2 \qquad (A \in \mathfrak{G}),$$

from which (2.5.9) becomes

$$\left\|\exp[g] \cdot \chi_{M^d_{\leq 1}\left(\sigma^{(n)}\right)}\right\|^2 = \prod_{k=1}^{n} \left(1 + \left\|g_{|G_{k,n}}\right\|^2\right) \qquad (n \in \mathbb{N}). \tag{2.5.14}$$

Now set $a_{k,n} := \left\|g_{|G_{k,n}}\right\|^2$ and $a := \sum_{k=1}^{n} a_{k,n} = \|g\|^2$. Because of (I) we may apply Lemma 2.5.1 to the right-hand side of (2.5.14), implying

$$\left\|\exp[g] \cdot \chi_{M^d_{\leq 1}\left(\sigma^{(n)}\right)}\right\|^2 \underset{n\to\infty}{\longrightarrow} e^{\|g\|^2}.$$

But this is equivalent to the assertion, because $\|\exp[g]\|^2 = e^{\|g\|^2}$. ■

Now define

$$h_{G_1,\ldots,G_n}(z,f,g) := \bigotimes_{k=1}^{n} h_{G_k}(z,f,g)$$
$$\left((G_1,\ldots,G_n) \in \Sigma(G), z \in \mathbb{C}, f,g \in L^2\left(G^{(d)}\right)\right), \tag{2.5.15}$$

where

$$h_A(z,f,g) := \exp\left[0_{|A}\right] + z\left(\exp\left[f_{|A}\right] + \exp\left[g_{|A}\right] - 2\exp\left[0_{|A}\right]\right)$$
$$\left(A \in \mathfrak{G}, z \in \mathbb{C}, f,g \in L^2\left(G^{(d)}\right)\right), \tag{2.5.16}$$

i.e.

$$h_A(z,f,g)(\varphi) = \begin{cases} 1 & \varphi = \mathcal{O} \\ z\exp[f](\varphi) + z\exp[g](\varphi) & \varphi \neq \mathcal{O} \end{cases}$$
$$\left(\varphi \in M\left(A^{(d)}\right), A \in \mathfrak{G}, z \in \mathbb{C}, f,g \in L^2\left(G^{(d)}\right)\right). \tag{2.5.17}$$

Lemma 2.5.3 *If $f,g \in L^2\left(G^{(d)}\right)$ and $\sigma^{(n)} := (G_{1,n},\ldots,G_{n,n}) \in \Sigma(G)$ for $n \in \mathbb{N}$, such that*

$$(I) \qquad \sup_{1\leq k\leq n}\left(\left\|f_{|G_{k,n}}\right\| + \left\|g_{|G_{k,n}}\right\|\right) \underset{n\to\infty}{\longrightarrow} 0,$$

then

$$h_{\sigma^{(n)}}(z,f,g) \underset{n\to\infty}{\longrightarrow} \exp[zf + zg] \qquad (z \in \mathbb{C}).$$

Proof: Fix $z \in \mathbb{C}$, $f, g \in L^2\big(G^{(d)}\big)$ and $\big(\sigma^{(n)}\big)_{n\in\mathbb{N}}$ for which (I) holds. We conclude from (2.5.17) that

$$h_A(z,f,g)(\varphi) = \exp\big[zf_{|A} + zg_{|A}\big]\,(\varphi) \qquad \Big(\varphi \in M_{\leq 1}\big(A^{(d)}\big), A \in \mathfrak{G}\Big), \tag{2.5.18}$$

i.e.

$$h_A(z,f,g)\cdot \chi_{M_{\leq 1}(A^{(d)})} = \exp\big[zf_{|A} + zg_{|A}\big]\cdot \chi_{M_{\leq 1}(A^{(d)})} \qquad (A \in \mathfrak{G}) \tag{2.5.19}$$

and therefore, using (2.5.6) and (2.5.15)

$$h_{\sigma^{(n)}}(z,f,g)\cdot \chi_{M^d_{\leq 1}(\sigma^{(n)})} = \exp[zf + zg]\cdot \chi_{M^d_{\leq 1}(\sigma^{(n)})} \qquad (n \in \mathbb{N}). \tag{2.5.20}$$

Furthermore we immediately have from definition that

$$|z| \leq \exp[\max\{1,|z|\}]\,(\varphi) \qquad \Big(\mathcal{O} \neq \varphi \in M\big(G^{(d)}\big), z \in \mathbb{C}\Big). \tag{2.5.21}$$

Hence, by multiplying respective sides of (1.4.7) and (2.5.21) and using (1.4.5),

$$|z|\,|\exp[f]\,(\varphi) + \exp[g]\,(\varphi)| \quad \leq \quad \exp[b]\,(\varphi) \qquad \Big(\mathcal{O} \neq \varphi \in M\big(G^{(d)}\big)\Big), \tag{2.5.22}$$

where

$$b := \max\{1,|z|\}\cdot\big(|f| + |g|\big). \tag{2.5.23}$$

But then (2.5.17) and (2.5.22) imply that

$$|h_A(z,f,g)| \leq \big|\exp\big[b_{|A}\big]\big| \qquad (A \in \mathfrak{G}), \tag{2.5.24}$$

and thus, using (2.5.15),

$$|h_{\sigma^{(n)}}(z,f,g)| \leq |\exp[b]| \qquad (n \in \mathbb{N}). \tag{2.5.25}$$

From (2.5.20), the triangular inequality and (2.5.25) we finally conclude

$$\begin{aligned}
&\Big\|\Big(h_{\sigma^{(n)}}(z,f,g) - \exp[zf+zg]\Big)\Big\| \\
&\qquad\qquad = \Big\|\Big(h_{\sigma^{(n)}}(z,f,g) - \exp[zf+zg]\Big)\cdot \chi^c_{M^d_{\leq 1}(\sigma^{(n)})}\Big\| \\
&\qquad\qquad \leq \Big\|h_{\sigma^{(n)}}(z,f,g)\cdot \chi^c_{M^d_{\leq 1}(\sigma^{(n)})}\Big\| + \Big\|\exp[zf+zg]\cdot \chi^c_{M^d_{\leq 1}(\sigma^{(n)})}\Big\| \\
&\qquad\qquad \leq \Big\|\exp[b]\cdot \chi^c_{M^d_{\leq 1}(\sigma^{(n)})}\Big\| + \Big\|\exp[zf+zg]\cdot \chi^c_{M^d_{\leq 1}(\sigma^{(n)})}\Big\| \qquad (n \in \mathbb{N}).
\end{aligned} \tag{2.5.26}$$

But by (2.5.23) and the triangular inequality

$$\big\|b_{|A}\big\|, \big\|zf_{|A} + zg_{|A}\big\| \leq \max\{1,|z|\}\cdot\big(\big\|f_{|A}\big\| + \big\|g_{|A}\big\|\big) \qquad (A \in \mathfrak{G}) \tag{2.5.27}$$

and thus, both terms on the right-hand side of (2.5.26) converge to zero because of (I) and by Lemma 2.5.2. This completes the proof. ■

2.6 Operators that map exponential vectors to exponential vectors

Starting point in this section is Proposition 1.4.5: For a Hilbert space H the set of exponential vectors $\exp[H] := \{\exp[f] : f \in H\}$ is linearily independent and total in $\Gamma(H)$. Thus, every $\psi \in \mathcal{E}(H) := \mathrm{Lin}\big(\exp[H]\big)$ has a unique representation $\psi = \sum_{i=1}^{n} c_i \exp[f_i]$ for scalars $c_i \in \mathbb{C}$ and vectors $f_i \in H$.

For Hilbert spaces H, H' an arbitrary, everywhere-defined map $T : H \to H'$ therefore defines a linear operator $U : \Gamma(H) \to \Gamma(H')$ with domain $\mathrm{dom}(U)$ containing exponential vectors by extension of

$$U \exp[f] := \exp[T(f)] \qquad (f \in H). \tag{2.6.1}$$

U is densely defined because, by linearity, $\mathcal{E}(H) \subseteq \mathrm{dom}(U)$ and

$$U\psi = \sum_{i=1}^{n} c_i \exp[T(f_i)] \qquad \left(\psi = \sum_{i=1}^{n} c_i \exp[f_i] \in \mathcal{E}(H)\right). \tag{2.6.2}$$

Lemma 1.4.7 tells us that, if T is a linear contraction or isometry, the same is true for U. In this case U is called second quantisation of T and denoted as $U = \Gamma(T)$.

But what if no other property of T other than it being an everywhere-defined map $T : H \to H'$ is known? Can properties like linearity, boundedness, isometry etc. of T be deduced by similar or even any kind of properties of U? Proposition 2.6.3 shows that if U is an isometry, so is T. Proposition 2.6.4 extends this result to isometries U given by

$$U \exp[f] = \gamma(f) \exp[T(f)] \qquad (f \in H) \tag{2.6.3}$$

for some scalar-valued and nowhere-vanishing function $\gamma : H \to \mathbb{C}$. This result is found in Liebscher [52, Prop B.4.2], the proof of which uses the theory of holomorphic functions for general Banach spaces. For $H = H'$ and U unitary it was already proven by Guichardet [44, Lemma 2.2] using more elementary methods. The proof presented here follows the ideas of Guichardet.

In proposition 2.6.6 we show, that if U is only bounded, but preserves regional subspaces in addition to exponential vectors, then T is linear and U is $\mathfrak{G}$-factorisable.

Remark 2.6.1 *Since for $c \in \mathbb{C}$ and $f \in H$*

$$c = \mathrm{Proj}_{\exp[0]} c \exp[f] \quad \textit{and} \quad f = \mathrm{Proj}_H \exp[f]\,, \tag{2.6.4}$$

both γ and T may be recovered from U given in (2.6.3) *by*

$$\gamma(f) = \mathrm{Proj}_{\exp[0]} U \exp[f] \quad \textit{and} \quad \gamma(f)T(f) = \mathrm{Proj}_H U \exp[f] \qquad (f \in H). \tag{2.6.5}$$

(2.6.1) *and* (2.6.3) *are therefore equivalent to $U\big(\exp[H]\big) \subseteq \exp[H']$ and $U\big(\exp[H]\big) \subseteq \mathbb{C}\exp[H']$, respectively.*

First we prove the result, that any everywhere-defined, scalar-product-preserving map ist linear.

Lemma 2.6.2 *Let H, H' be Hilbert spaces, both scalar products taking values in $K \in \{\mathbb{R}, \mathbb{C}\}$ and $T : H \to H'$ an everywhere-defined, scalar-product-preserving map. Then T is K-linear, hence isometric.*

Proof: Fix $g, h \in H, \lambda \in K$ and define $u := T(\lambda g + h) - \big(\lambda T(g) + T(h)\big)$. We are to show that $u = 0$. Since T is scalar-product-preserving we find

$$\begin{aligned}\langle T(f), T(\lambda g + h)\rangle_{H'} &= \langle f, \lambda g + h\rangle_H = \lambda \langle f, g\rangle_H + \langle f, h\rangle_H \\ &= \lambda \langle T(f), T(g)\rangle_{H'} + \langle T(f), T(h)\rangle_{H'} = \langle T(f), \lambda T(g) + T(h)\rangle_{H'} \\ &\qquad (f \in H) \quad (2.6.6)\end{aligned}$$

leading to

$$\langle T(f), u\rangle_{H'} = 0 \qquad (f \in H). \tag{2.6.7}$$

Replacing in this identity f with $\lambda g + h$, replacing f with g and multiplying both sides with $\overline{\lambda}$ and replacing f with h yield

$$\langle T(\lambda g + h), u\rangle_{H'} = 0, \qquad \langle \lambda T(g), u\rangle_{H'} = 0 \qquad \text{and} \qquad \langle T(h), u\rangle_{H'} = 0,$$

respectively. Subtracting the second and the third equation from the first finally gives $\langle u, u\rangle_{H'} = 0$, i.e. $u = T(\lambda g + h) - \big(\lambda T(g) + T(h)\big) = 0$, showing linearity, because $g, h \in H$ and $\lambda \in K$ were arbitrary. ■

Proposition 2.6.3 *Let H, H' be Hilbert spaces and $U : \Gamma(H) \to \Gamma(H')$ an isometry such that $U\big(\exp[H]\big) \subseteq \exp[H']$. Then there exists an isometry $T : H \to H'$ such that $U = \Gamma(T)$.*

Proof: According to remark 2.6.1 there exists an everywhere-defined map $T : H \to H'$ such that

$$U \exp[f] = \exp[T(f)] \qquad (f \in H). \tag{2.6.8}$$

We are to show that T is indeed isometric. As U is isometric, we have using Remark 1.4.3

$$\begin{aligned}e^{\langle T(f), T(g)\rangle_{H'}} &= \langle \exp[T(f)]\,, \exp[T(g)]\rangle_{\Gamma(H')} = \langle U\exp[f]\,, U\exp[g]\rangle_{\Gamma(H')} \\ &= \langle \exp[f]\,, \exp[g]\rangle_{\Gamma(H)} = e^{\langle f, g\rangle_H} \qquad (f, g \in H) \quad (2.6.9)\end{aligned}$$

which implies

$$\langle T(f), T(g)\rangle_{H'} - \langle f, g\rangle_H \in 2\pi i\mathbb{Z} \qquad (f, g \in H) \tag{2.6.10}$$

and

$$\operatorname{Re}\langle T(f), T(g)\rangle_{H'} = \operatorname{Re}\langle f, g\rangle_H \qquad (f, g \in H). \tag{2.6.11}$$

Considering H and H' as real Hilbert spaces with scalar products $\operatorname{Re}\langle\cdot,\cdot\rangle_H$ and $\operatorname{Re}\langle\cdot,\cdot\rangle_{H'}$, respectively, (2.6.11) and Lemma 2.6.2 tell us that T is $\mathbb{R}$-linear. Therefore, using (2.6.10),

$$\lambda\big(\langle T(f), T(g)\rangle_{H'} - \langle f, g\rangle_H\big) = \langle T(f), T(\lambda g)\rangle_{H'} - \langle f, \lambda g\rangle_H \in 2\pi i\mathbb{Z} \qquad (f, g \in H, \lambda \in \mathbb{R}) \tag{2.6.12}$$

which shows that, in fact,

$$\langle T(f), T(g)\rangle_{H'} - \langle f, g\rangle_H = 0 \qquad (f, g \in H). \tag{2.6.13}$$

By Lemma 2.6.2 again, T is isometric. ∎

Proposition 2.6.4 *Let H, H' be Hilbert spaces and $U : \Gamma(H) \to \Gamma(H')$ an isometry such that $U\big(\exp[H]\big) \subseteq \mathbb{C}\exp[H']$. Then there exists an isometry $T : H \to H'$, a vector $h \in H'$ and a constant $c \in \mathbb{C}, |c| = 1$, such that*

$$U = c \cdot \mathcal{W}(h)\Gamma(T). \tag{2.6.14}$$

Proof: According to remark 2.6.1 and because U is isometric, there exist everywhere-defined maps $R : H \to H'$ and $\gamma : H \to \mathbb{C}$ such that

$$U\exp[f] = \gamma(f) \cdot \exp[R(f)], \quad |\gamma(f)|^2 = e^{\|f\|_H^2 - \|R(f)\|_{H'}^2} \qquad (f \in H). \tag{2.6.15}$$

Set

$$\tilde{U} := \frac{1}{c} \cdot \mathcal{W}(-h)\, U, \quad \text{where} \quad c := \frac{\gamma(0)}{|\gamma(0)|} \quad \text{and} \quad h := R(0). \tag{2.6.16}$$

Then $U = c \cdot \mathcal{W}(h)\,\tilde{U}$, with $|c| = 1$ and $h \in H'$ and we are left to show that $\tilde{U} = \Gamma(T)$ for some isometry $T : H \to H'$. To see this, we first observe that $\tilde{U}$ is again isometric and maps exponential vectors to multiples of exponential vectors because both U and $\mathcal{W}(-h)$ do, i.e. there exist everywhere-defined maps $\tilde{R} : H \to H'$ and $\tilde{\gamma} : H \to \mathbb{C}$ such that

$$\tilde{U}\exp[f] = \tilde{\gamma}(f) \cdot \exp\Big[\tilde{R}(f)\Big] \qquad (f \in H). \tag{2.6.17}$$

Furthermore $\tilde{U}$ preserves the vacuum because by (2.6.15) and (2.6.16)

$$U\exp[0] = \gamma(0) \cdot \exp[R(0)] = c \cdot |\gamma(0)| \cdot \exp[h] = c \cdot e^{-\frac{1}{2}\|h\|_{H'}^2} \cdot \exp[h] \tag{2.6.18}$$

and therefore

$$\tilde{U}\exp[0] = \frac{1}{c} \cdot \mathcal{W}(-h)\, U\exp[0] = e^{-\frac{1}{2}\|h\|_{H'}^2} \cdot \mathcal{W}(-h)\exp[h] = \exp[0]\,. \tag{2.6.19}$$

But then

$$\begin{aligned}\tilde{\gamma}(f) &= \left\langle \exp[0]\,,\tilde{\gamma}(f)\cdot\exp\left[\tilde{R}(f)\right]\right\rangle_{\Gamma(H')} \\ &= \left\langle \tilde{U}\exp[0]\,,\tilde{U}\exp[f]\right\rangle_{\Gamma(H')} = \langle \exp[0]\,,\exp[f]\rangle_{\Gamma(H)} = 1 \quad (f\in H), \end{aligned} \tag{2.6.20}$$

i.e.

$$\tilde{U}\exp[f] = \exp\left[\tilde{R}(f)\right] \quad (f\in H), \tag{2.6.21}$$

where $T := \tilde{R}$ is isometric by proposition 2.6.3. This completes the proof. ■

Remark 2.6.5 *Operators of the just derived type $U = c\cdot\mathcal{W}(h)\Gamma(T)$ act on exponential vectors as*

$$U\exp[f] = c\cdot e^{-\frac{1}{2}\|h\|^2-\langle h,f\rangle}\exp[Tf+h] = \gamma_h(f)\cdot\exp[Tf+h] \qquad (f\in H). \tag{2.6.22}$$

This means that the action of U on coherent particle systems is described by independently performing a so-called rigid motion ("rotation" plus translation) on the one-particle subspaces, followed by a "re-isometrisation" through multipliclation with $\gamma_h(f)$.

Specialising to $H = L^2\big(G^{(d_1)}\big)$ and $H' = L^2\big(G^{(d_2)}\big)$ we can give a similar result to proposition 2.6.3, even if the Fock space operator B is only bounded and not isometric. We still get a linear operator T on the one-particle subspace. But we have to require T to preserve regional subspaces, making B in turn $\mathfrak{G}$-factorisable.

Recall that for bounded $B : \mathcal{M}\big(G^{(d_1)}\big) \to \mathcal{M}\big(G^{(d_2)}\big)$, its (local) restriction to $\mathcal{M}\big(A^{(d_1)}\big)$ is given on exponential vectors through

$$B_{|A}\exp\big[f_{|A}\big] = \big(B\exp[\chi_A f]\big)_{|A} \qquad \Big(f\in L^2\big(G^{(d_1)}\big)\Big). \tag{2.6.23}$$

Proposition 2.6.6 *For $d_1, d_2 \in \mathbb{N}$ let $B : \mathcal{M}\big(G^{(d_1)}\big) \to \mathcal{M}\big(G^{(d_2)}\big)$ be a bounded operator that preserves both regional subspaces and exponential vectors, i.e. there exists an everywhere-defined map $T : L^2\big(G^{(d_1)}\big) \to L^2\big(G^{(d_2)}\big)$ such that*

$$B\exp[f] = \exp[T(f)] \quad \text{and} \quad T(\chi_A h) = \chi_A T(h) \qquad \Big(f,h\in L^2\big(G^{(d_1)}\big), A\in\mathfrak{G}\Big). \tag{2.6.24}$$

Then

$$B = \bigotimes_{k=1}^{n} B_{|G_k} \qquad \big((G_1,\dots,G_n)\in\Sigma(G)\big) \tag{2.6.25}$$

and T is linear.

Proof: Using (2.6.23), (2.6.24) and the fact that $\exp[f]_{|A} = \exp\left[f_{|A}\right]$,

$$B_{|A}\exp\left[f_{|A}\right] = \left(B\exp[\chi_A f]\right)_{|A} = \exp[\chi_A T(f)]_{|A} = \exp\left[T(f)_{|A}\right]$$
$$\Big(f \in L^2\big(G^{(d_1)}\big), A \in \mathfrak{G}\Big). \quad (2.6.26)$$

This implies

$$\left(\bigotimes_{k=1}^n B_{|G_k}\right)\left(\bigotimes_{k=1}^n \exp\left[f_{|G_k}\right]\right) = \bigotimes_{k=1}^n \exp\left[T(f)_{|G_k}\right] = \exp[T(f)] = B\exp[f]$$
$$\Big(f \in L^2\big(G^{(d_1)}\big), (G_1, \ldots, G_n) \in \Sigma(G)\Big), \quad (2.6.27)$$

thus showing (2.6.25), because it is enough to take exponential vectors as arguments.

We well show linearity of T. To this end, define $\big($see(2.5.15) and (2.5.16) for details$\big)$

$$h_{G_1,\ldots,G_n}(z, f, g) := \bigotimes_{k=1}^n h_{G_k}(z, f, g)$$
$$\Big(n \in \mathbb{N}, (G_1, \ldots, G_n) \in \Sigma(G), z \in \mathbb{C}, f, g \in L^2\big(G^{(d)}\big)\Big), \quad (2.6.28)$$

where

$$h_A(z, f, g) := \exp\left[0_{|A}\right] + z\Big(\exp\left[f_{|A}\right] + \exp\left[g_{|A}\right] - 2\exp\left[0_{|A}\right]\Big)$$
$$\Big(A \in \mathfrak{G}, z \in \mathbb{C}, f, g \in L^2\big(G^{(d)}\big)\Big). \quad (2.6.29)$$

In (2.6.24) choosing A to be the empty set shows that

$$T(0) = 0. \quad (2.6.30)$$

Now fix $f, g \in L^2\big(G^{(d)}\big)$ and $z \in \mathbb{C}$. We conclude from (2.6.29), linearity of $B_{|A}$, (2.6.26) and (2.6.30)

$$\begin{aligned}
& B_{|A}h_A(z, f, g) \\
= \; & B_{|A}\exp\left[0_{|A}\right] + z\Big(B_{|A}\exp\left[f_{|A}\right] + B_{|A}\exp\left[g_{|A}\right] - 2B_{|A}\exp\left[0_{|A}\right]\Big) \\
= \; & \exp\left[T(0)_{|A}\right] + z\Big(\exp\left[T(f)_{|A}\right] + \exp\left[T(g)_{|A}\right] - 2\exp\left[T(0)_{|A}\right]\Big) \\
= \; & h_A\big(z, T(f), T(g)\big) \qquad (A \in \mathfrak{G})
\end{aligned}$$

and therefore, by (2.6.25) and (2.6.28),

$$Bh_{\sigma^{(n)}}(z, f, g) = \bigotimes_{k=1}^n B_{|G_k}h_{G_k}(z, f, g) = \bigotimes_{k=1}^n h_{G_k}\big(z, T(f), T(g)\big)$$
$$= h_{\sigma^{(n)}}\big(z, T(f), T(g)\big) \qquad \big(\sigma^{(n)} = (G_1, \ldots, G_n) \in \Sigma(G)\big). \quad (2.6.31)$$

Then Lemma 2.5.3 tells us that

$$\exp[zf + zg] = \lim_{n\to\infty} h_{\sigma^{(n)}_{f,g}}(z, f, g), \tag{2.6.32}$$

and

$$\exp[zT(f) + zT(g)] = \lim_{n\to\infty} h_{\sigma^{(n)}_{f,g}}\big(z, T(f), T(g)\big) \tag{2.6.33}$$

where $\sigma^{(n)}_{f,g} = (G_{1,n}, \ldots, G_{n,n}) \in \Sigma(G)$, such that

$$\sup_{1\le k\le n}\Big(\big\|f_{|G_{k,n}}\big\| + \big\|T(f)_{|G_{k,n}}\big\| + \big\|g_{|G_{k,n}}\big\| + \big\|T(g)_{|G_{k,n}}\big\|\Big) \underset{n\to\infty}{\longrightarrow} 0. \tag{2.6.34}$$

From (2.6.24), (2.6.32), boundedness of B, (2.6.31) and (2.6.33) we thus have

$$\begin{aligned}\exp[T(zf + zg)] = B\exp[zf + zg] &= \lim_{n\to\infty} Bh_{\sigma^{(n)}_{f,g}}(z, f, g)\\ &= \lim_{n\to\infty} h_{\sigma^{(n)}_{f,g}}\big(z, T(f), T(g)\big) = \exp[zT(f) + zT(g)]\,.\end{aligned} \tag{2.6.35}$$

Using Lemma 1.4.4 (the map $\exp[\cdot]$ is one-to-one), this becomes

$$T(zf + zg) = zT(f) + zT(g) \tag{2.6.36}$$

implying

$$T(zf) = zT(f) \qquad \text{and} \qquad T(f + g) = T(f) + T(g). \tag{2.6.37}$$

Since $f, g \in L^2\big(G^{(d)}\big)$ and $z \in \mathbb{C}$ were arbitrary this shows linearity of T. ■

3 Beam Splittings and their Application

Notation: Remembering $L^2\big(G^{(d)}\big) \cong L^2\big(G, \mathbb{C}^d\big)$, we will write

$$f(x) := \big(f_1(x), \ldots, f_d(x)\big) \in \mathbb{C}^d \qquad \Big(f \in L^2\big(G^{(d)}\big), x \in G\Big). \tag{3.0.1}$$

Denote with

$$\mathbb{M}^{m\times n} := \mathbb{M}^{m\times n}(G) := \mathbb{M}\big(G, \mathbb{C}^{m\times n}\big)$$

the set of measurable maps from G to $\mathbb{C}^{m\times n}$. For $n = 1$ or $m = n = 1$ we will simply write $\mathbb{M}^m$ or $\mathbb{M}$ respectively.

For $\alpha \in \mathbb{M}$ and $f \in \mathbb{M}^m$ also denote with $\alpha f \in \mathbb{M}^m$ the function defined through

$$(\alpha f)(x) := \alpha(x) \cdot f(x) \qquad (x \in G). \tag{3.0.2}$$

In particular, for constant $f(x) \equiv c \in \mathbb{C}^m$, we will write αc.

3.1 Introduction

So far, at least the following kinds of beam splittings have been treated:

1. Let an isometry $\mathcal{V}_{\alpha,\beta} : \mathcal{M}\big(G\big) \to \mathcal{M}^{\otimes 2}(G)$ be defined through

   $$\mathcal{V}_{\alpha,\beta} \exp[f] := \exp[\alpha f] \otimes \exp[\beta f] \qquad \big(f \in L^2(G)\big) \tag{3.1.1}$$

   for $\alpha, \beta \in \mathbb{M}$ such that $|\alpha|^2 + |\beta|^2 \equiv 1$. Then $\mathcal{V}_{\alpha,\beta}$ represents the splitting of one beam of input into two beams of output with splitting rates α and β. It is sometimes called an attenuation. For details see [22, 24, 23] and, in terms of transition expectations and in relation to quantum Markov chains [1, 36, 63]).

2. As a generalisation of $\mathcal{V}_{\alpha,\beta}$ in 1. define the isometry $\mathcal{V}_{S,T} : \Gamma(H) \to \Gamma(H) \otimes \Gamma(H)$ through

   $$\mathcal{V}_{S,T} \exp[f] := \exp[Sf] \otimes \exp[Tf] \qquad (f \in H) \tag{3.1.2}$$

   for bounded operators S, T on H such that $S^*S + T^*T = \mathbb{1}_H$. For this approach see [52] and its application in quantum teleportation [30, 31].

3. Define the unitary operator $\mathcal{V} : \mathcal{M}^{\otimes 2}(G) \to \mathcal{M}^{\otimes 2}(G)$ via

$$\mathcal{V} \exp[f_1] \otimes \exp[f_2] := \exp[\alpha_1 f_1 + \alpha_2 f_2] \otimes \exp[\beta_1 f_1 + \beta_2 f_2] \qquad \big(f_1, f_2 \in L^2(G)\big) \quad (3.1.3)$$

for suitable $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{M}$. Here $\mathcal{V}$ describes the splitting of two beams of input and their recombination into again two beams of output. Thus, it could (or should) in fact be called a beam mixing. For the use of $\mathcal{V}$ in the construction of quantum logical gates see [41]. As an example of a more general kind of "exchange operators" $\mathcal{V}$ is called "independent exchange" in [25] (see also section 3.4 for a brief introduction).

Let us take a closer look at $\mathcal{V}$ from item 3. above. We have the isomorphic identification:

$$\mathcal{M}^{\otimes d}(G) \cong \mathcal{M}\big(G^{(d)}\big) \qquad \text{and} \qquad \bigotimes_{i=1}^{d} \exp[f_i] \cong \exp[f]\,, \qquad (3.1.4)$$

where $f = (f_1, \ldots, f_d) \in L^2\big(G^{(d)}\big)$. In (3.1.3) we may therefore equivalently write $\mathcal{V} = \Gamma(B)$, i.e

$$\mathcal{V} \exp[f] = \exp[Bf] \qquad \Big(f \in L^2\big(G^{(2)}\big)\Big),$$

for $B : L^2\big(G^{(2)}\big) \to L^2\big(G^{(2)}\big)$ acting on $f = (f_1, f_2) \in L^2\big(G^{(2)}\big)$ according to

$$(Bf)(x) := \begin{pmatrix} \alpha_1(x) & \alpha_2(x) \\ \beta_1(x) & \beta_2(x) \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \qquad (x \in G). \qquad (3.1.5)$$

Thus B is defined by pointwise action of the matrix-valued function $b = \left(\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{smallmatrix}\right) \in \mathbb{M}^{2\times 2}$ on the right-hand side.

We want to generalise this idea to an arbitrary number of beams of in- and output. After a brief excursion to the theory of operators of matrix multiplication in section 3.2 we will return to defining and characterising beam splittings in section 3.3 followed by aplications and further insight in sections 3.4 and 3.5.

3.2 Operators of Matrix Multiplication

Throughout this section $d_1, d_2 \in \mathbb{N}$ will be fixed. For basic results and other applications of operators of matrix multiplication see [6], [46] and [45].

Any measurable matrix-valued function $b \in \mathbb{M}^{d_2 \times d_1}$ defines a linear operator O_b :

$L^2\big(G^{(d_1)}\big) \to L^2\big(G^{(d_2)}\big)$ by acting pointwise, i.e.

$$(O_b f)(x) := b(x)f(x) = \begin{pmatrix} b_{1,1}(x) & \cdots & b_{1,d_1}(x) \\ \vdots & \ddots & \vdots \\ b_{d_2,1}(x) & \cdots & b_{d_2,d_1}(x) \end{pmatrix} \begin{pmatrix} f_1(x) \\ \vdots \\ f_{d_1}(x) \end{pmatrix}$$
$$\big(f \in \mathrm{dom}(O_b)\,, x \in G\big). \quad (3.2.1)$$

Its maximum domain is given by $f \in \mathrm{dom}(O_b) \subseteq L^2\big(G^{(d_1)}\big)$ if and only if

$$\|O_b f\|^2_{L^2(G^{(d_2)})} = \int_G \|b(x)f(x)\|^2_{\mathbb{C}^{d_2}}\, d\mu(x) < \infty. \quad (3.2.2)$$

Remark 3.2.1 *For an operator to be of the kind O_b it is enough if* (3.2.1) *holds for μ-a.e. $x \in G$. In addition, $O_b = O_{b'}$ if and only if $b = b'$ μ-a.e.*

Definition 3.2.2 *The operator O_b defined as in* (3.2.1) *is called operator of matrix multiplication associated with $b = \big(b_{i,j}\big) \in \mathbb{M}^{d_2\times d_1}$.*

We will summarise some of the properties of operators of matrix multiplication in Lemma 3.2.3. Therein

$$\|b\|_\infty := \operatorname*{ess\,sup}_{x\in G} \|b(x)\| \le \infty \qquad \big(b \in \mathbb{M}^{d_2\times d_1}\big)\,, \quad (3.2.3)$$

where $\|b(x)\|$ denotes the (spectral) norm of the linear operator represented by the matrix $b(x)$, i.e.

$$\|D\| := \sup\big\{\|Dy\|_{\mathbb{C}^{d_2}} : y \in \mathbb{C}^{d_1}, \|y\|_{\mathbb{C}^{d_1}} = 1\big\} \qquad \big(D \in \mathbb{C}^{d_2\times d_1}\big)\,. \quad (3.2.4)$$

Lemma 3.2.3 *Let O_b be an operator of matrix-multiplication. Then*

(1) O_b is bounded if and only if all the components $b_{i,j}$ are essentially bounded. We have $\|O_b\| = \|b\|_\infty$ for all $b \in \mathbb{M}^{d_2\times d_1}$.

(2) $O_b^* = O_{b^*}$, i.e.

$$(O_b^* f)(x) = b^*(x)f(x) \qquad \Big(f \in L^2\big(G^{(d_1)}\big), \mu\text{-a.e. } x \in G\Big), \quad (3.2.5)$$

where $b^*(x)$ denotes the adjoint matrix (conjugate transpose) of $b(x)$.

(3) $O_b^* O_b = \mathbb{1}_{L^2(G^{(d_1)})}$ if and only if $b^*(x)b(x) = \mathbb{1}_{\mathbb{C}^{d_1}}$ for μ-a.e. $x \in G$.

(4) O_b is isometric if and only if the matrices $b(x)$ are isometric for μ-a.e. $x \in G$.

(5) O_b and O_α commute for all $\alpha \in \mathbb{M}$ in the sense of

$$O_b(\alpha f) = \alpha O_b f \qquad \big(\alpha \in \mathbb{M}, f, \alpha f \in \operatorname{dom}(O_b)\big). \tag{3.2.6}$$

Proof:

(1) First we will show $\|O_b\| = \|b\|_\infty$. Fix $b \in \mathbb{M}^{d_2 \times d_1}$. For $f \in \operatorname{dom}(O_b)$

$$\begin{aligned}
\|O_b f\|^2_{L^2(G^{(d_2)})} &= \int \|b(x)f(x)\|^2_{\mathbb{C}^{d_2}}\, d\mu(x) \\
&\le \int \|b(x)\|^2 \|f(x)\|^2_{\mathbb{C}^{d_1}}\, d\mu(x) \qquad \text{by (3.2.4)} \\
&\le \int \|b\|^2_\infty \|f(x)\|^2_{\mathbb{C}^{d_1}}\, d\mu(x) \qquad \text{by (3.2.3)} \\
&= \|b\|^2_\infty \|f\|^2_{L^2(G^{(d_1)})}
\end{aligned}$$

and therefore

$$\|O_b\| \le \|b\|_\infty. \tag{3.2.7}$$

Now fix $c < \|b\|_\infty$. Then, by (3.2.3) and because μ is σ-finite (G is separable and μ is locally finite), there exists $A \in \mathfrak{G}$ such that $0 < \mu(A) < \infty$ and $\|b(x)\| \ge c$ for μ-a.e. $x \in A$. Hence, by (3.2.4), there exists $g \in \mathbb{M}^{d_1}$ such that

$$\|g(x)\|_{\mathbb{C}^{d_1}} = 1 \quad \text{and} \quad \|b(x)g(x)\|_{\mathbb{C}^{d_2}} \ge c \qquad (\mu\text{-a.e. } x \in A). \tag{3.2.8}$$

Setting $f := \frac{1}{\mu(A)}\chi_A g$ we have

$$\|f\|^2_{L^2(G^{(d_1)})} = \frac{1}{\mu(A)} \int_A \|g(x)\|^2_{\mathbb{C}^{d_1}}\, d\mu(x) = 1$$

and, using (3.2.8),

$$\|O_b f\|^2_{L^2(G^{(d_2)})} = \frac{1}{\mu(A)} \int_A \|b(x)g(x)\|^2_{\mathbb{C}^{d_2}}\, d\mu(x) \ge c^2,$$

showing $\|O_b\| \ge c$ for any $c < \|b\|_\infty$ because it was arbitrary. Together with (3.2.7) this proves $\|O_b\| = \|b\|_\infty$.

We are left to show that $\|b\|_\infty < \infty$ if and only if all the components $b_{i,j}$ are essentially bounded. It is well-known, that the maximum and the spectral norm for matrices are equivalent (see for instance [5, p. 628]). In particular:

$$\|C\|_{\max} \le \|C\| \le \sqrt{d_1 \cdot d_2} \cdot \|C\|_{\max} \qquad \big(C \in \mathbb{C}^{d_2 \times d_1}\big),$$

where

$$\|C\|_{\max} := \max_{i,j} |C_{i,j}| \qquad \big(C \in \mathbb{C}^{d_2 \times d_1}\big).$$

Setting $C = b(x)$ and passing over to the essential supremum yields

$$\operatorname*{ess\,sup}_{x\in G} \max_{i,j} |b_{i,j}(x)| \leq \|b\|_\infty \leq \sqrt{d_1 \cdot d_2} \cdot \operatorname*{ess\,sup}_{x\in G} \max_{i,j} |b_{i,j}(x)| \,.$$

This completes the proof, because the order of maximum and essential supremum may be interchanged, i.e.

$$\max_{i,j} \|b_{i,j}\|_\infty \leq \|b\|_\infty \leq \sqrt{d_1 \cdot d_2} \cdot \max_{i,j} \|b_{i,j}\|_\infty \,.$$

(2) Since the conjugate transpose $b^*(x)$ of a matrix indeed represents the adjoint operator given by the matrix $b(x)$ we have for all $f \in \operatorname{dom}(O_{b^*})$ and $g \in \operatorname{dom}(O_b)$:

$$\begin{aligned} \langle O_{b^*} f, g\rangle_{L^2(G^{(d_1)})} &= \int \langle b^*(x) f(x), g(x)\rangle_{\mathbb{C}^{d_1}} \; d\mu(x) \\ &= \int \langle f(x), b(x) g(x)\rangle_{\mathbb{C}^{d_2}} \; d\mu(x) \\ &= \langle f, O_b g\rangle_{L^2(G^{(d_2)})} \,. \end{aligned}$$

Because the adjoint of an operator is unique, this implies $O_b^* = O_{b^*}$.

(3) $(O_b^* O_b f)(x) = b^*(x) b(x) f(x)$ for all $f \in L^2\left(G^{(d_1)}\right)$ and μ-a.e. $x \in G$.

(4) If O_b is isometric, (3) implies the assertion. Conversely, if the matrices $b(x)$ are isometric for μ-a.e. $x \in G$, O_b being isometric follows from

$$\begin{aligned} \|O_b f\|^2_{L^2(G^{(d_2)})} &= \int \|b(x) f(x)\|^2_{\mathbb{C}^{d_2}} \; d\mu(x) = \int \|f(x)\|^2_{\mathbb{C}^{d_1}} \; d\mu(x) \\ &= \|f\|^2_{L^2(G^{(d_1)})} \qquad \Big(f \in L^2\big(G^{(d_1)}\big)\Big). \end{aligned}$$

(5) By linearity of $b(x)$

$$\begin{aligned} \big(O_b(\alpha f)\big)(x) = b(x)\big(\alpha(x)\cdot f(x)\big) &= \alpha(x)\cdot b(x) f(x) = \alpha(x)\cdot (O_b f)(x) \\ &= (\alpha O_b f)(x) \qquad (\alpha \in \mathbb{M}, f, \alpha f \in \operatorname{dom}(O_b)\,, x \in G). \end{aligned}$$

■

Choosing $\alpha = \chi_A$ in (3.2.6) shows that O_b and $O_A = O_{\chi_A}$ commute for all $A \in \mathfrak{G}$, i.e.

$$O_b(\chi_A f) = \chi_A O_b f \qquad \big(f \in \operatorname{dom}(O_b)\,, A \in \mathfrak{G}\big). \tag{3.2.9}$$

Equation (3.2.9) means that O_b maps $L^2\big(A^{(d_1)}\big)$ into $L^2\big(A^{(d_2)}\big)$, if these are considered as regional subspaces of $L^2\big(G^{(d_1)}\big)$ and $L^2\big(G^{(d_2)}\big)$, repectively. This leads to the following

Definition 3.2.4 *If an operator $T : L^2\big(G^{(d_1)}\big) \to L^2\big(G^{(d_2)}\big)$ preserves regional subspaces, i.e.*

$$T(\chi_A f) = \chi_A T f \qquad \big(f \in \mathrm{dom}(T)\,, A \in \mathfrak{G}\big), \tag{3.2.10}$$

it will be called local.

Thus, by (3.2.9),

Corollary 3.2.5 *O_b is local.* ■

We now want to find a criterion by which to tell, whether for a given and bounded B there holds $B = O_b$ and how to determine $b = (b_{i,j})$ in this case.

Let $(e_i)_{i=1}^{d_1}$ be the canonical orthonormal basis in $\mathbb{C}^{d_1}$, i.e. the i-th component of e_i is 1 and all the others are 0. As a necessary condition for B to be of the kind $B = O_b$ we then have for the i-th column of b

$$b(x)e_i = b(x)\big((\chi_G e_i)(x)\big) = B(\chi_G e_i)(x) \qquad (\mu\text{-a.e. } x \in G, 1 \le i \le d_1). \tag{3.2.11}$$

Example 3.2.6 *$d_1 = d_2 = 2$. Then $\chi_G e_1 = \binom{\chi_G}{0}, \chi_G e_2 = \binom{0}{\chi_G}$. Define $\binom{\alpha_i}{\beta_i} := B(\chi_G e_i)$. Then $b(x)e_i = \binom{\alpha_i(x)}{\beta_i(x)}$, i.e. $b(x) = \begin{pmatrix} \alpha_1(x) & \alpha_2(x) \\ \beta_1(x) & \beta_2(x) \end{pmatrix}$.* ■

But there is a problem with (3.2.11) and example 3.2.6, namely

$$\|\chi_A e_i\|_{L^2(G^{(d_1)})} = \|\chi_A\|_{L^2(G)} = \mu(A) \qquad (1 \le i \le d_1, A \in \mathfrak{G}), \tag{3.2.12}$$

i.e. $\chi_G e_i \in L^2\big(G^{(d_1)}\big)$ if and only if μ is finite. And only in this case the right-hand side of (3.2.11) makes sense. But what if μ is only locally finite?

Theorem 3.2.7 *For a bounded operator the following three conditions are equivalent:*

(1) $B = O_b$ for some $b \in \mathbb{M}^{d_2 \times d_1}$.

(2) $B(\alpha f) = \alpha B f \qquad \Big(\alpha \in \mathbb{M}, f, \alpha f \in L^2\big(G^{(d_1)}\big)\Big)$.

(3) B is local.

In this case the i-th column of the matrix-valued function $b \in \mathbb{M}^{d_2 \times d_1}$ is given piecewise by

$$b(x)e_i = B\big(\chi_{G_j} e_i\big)(x) \qquad (j \in \mathbb{N}, \mu\text{-a.e. } x \in G_j), \tag{3.2.13}$$

where $(e_i)_{i=1}^{d_1}$ is the canonical ONB in $\mathbb{C}^{d_1}$ and $(G_j)_{j \in \mathbb{N}}$ is an arbitrary decomposition of G into disjoint and bounded $G_j \in \mathfrak{G}$.

Proof: (1) implies (2): Property (5) from Lemma 3.2.3.

(2) implies (3): Take $\alpha = \chi_A$ and Definition 3.2.4.

(3) implies (1): Fix an arbitrary decomposition of G into disjoint and bounded $G_j \in \mathfrak{G}, j \in \mathbb{N}$, and define the i-th column of $b \in \mathbb{M}^{d_2 \times d_1}$ piecewise for all $j \in \mathbb{N}$ and $x \in G_j$ by

$$b(x)e_i := B\big(\chi_{G_j} e_i\big)(x) \qquad (j \in \mathbb{N}, x \in G_j, 1 \le i \le d_1). \tag{3.2.14}$$

Then, by definition of $(G_j)_{j \in \mathbb{N}}$, assumption (3), (3.2.14) and Lemma 3.2.5

$$\chi_{G_j} B(\chi_A e_i) = \chi_A B\left(\chi_{G_j} e_i\right) = \chi_A O_b\left(\chi_{G_j} e_i\right) = \chi_{G_j} O_b(\chi_A e_i)$$
$$(1 \le i \le d_1, j \in \mathbb{N}, \text{ bounded } A \in \mathfrak{G}) \tag{3.2.15}$$

i.e.

$$B(\chi_A e_i) = O_b(\chi_A e_i) \qquad (1 \le i \le d_1, \text{ bounded } A \in \mathfrak{G}), \tag{3.2.16}$$

which shows that $\chi_A e_i \in \operatorname{dom}(O_b)$ for all $1 \le i \le d_1$ and bounded $A \in \mathfrak{G}$. But B is bounded and $E := \{\chi_A e_i : 1 \le i \le d_1, \text{ bounded } A \in \mathfrak{G}\}$ is total in $L^2\big(G^{(d_1)}\big)$. Hence B is the unique bounded extension of O_b from E to the whole of $L^2\big(G^{(d_1)}\big)$. This shows (1). We are left to prove that (3.2.13) holds independent of the particular decomposition $(G_j)_{j \in \mathbb{N}}$. Now, for another decomposition $(G'_k)_{k \in \mathbb{N}}$, define

$$b'(x)e_i = B\big(\chi_{G'_k} e_i\big)(x) \qquad (k \in \mathbb{N}, x \in G'_k, 1 \le i \le d_1). \tag{3.2.17}$$

Then we have, using the assumption (3),

$$\chi_{G_j} \chi_{G'_k} B\big(\chi_{G_j} e_i\big) = \chi_{G_j} \chi_{G'_k} B\big(\chi_{G'_k} e_i\big) \qquad (j, k \in \mathbb{N}, 1 \le i \le d_1). \tag{3.2.18}$$

By (3.2.14) and (3.2.17), this shows $b = b'$ μ-a.e. and hence b is μ-a.e. independent of $(G_j)_{j \in \mathbb{N}}$. ∎

3.3 Beam Splittings and their Characterisation

We now return to the original problem of defining and characterising beam splittings with an arbitrary number of in- and output beams.

Definition 3.3.1 *Let $b \in \mathbb{M}^{d_2 \times d_1}$ be such that $b(x)$ is isometric for μ-a.e. $x \in G$. Then $\mathcal{V}_b := \Gamma(O_b) : \mathcal{M}\big(G^{(d_1)}\big) \to \mathcal{M}\big(G^{(d_2)}\big)$ given through*

$$\mathcal{V}_b \exp[f] := \Gamma(O_b) \exp[f] = \exp[O_b f] \qquad \Big(f \in L^2\big(G^{(d_1)}\big)\Big) \tag{3.3.1}$$

is called beam splitting with d_1 beams of input, d_2 beams of output and splitting rate b.

Remark 3.3.2 *The name is justified by the identification*

$$\mathcal{M}\big(G^{(d_i)}\big) \cong \mathcal{M}^{\otimes d_i}(G). \tag{3.3.2}$$

Definition 3.3.1 and the results from the last subsection imply

Theorem 3.3.3 *An operator $V : \mathcal{M}\big(G^{(d_1)}\big) \to \mathcal{M}\big(G^{(d_2)}\big)$ is a beam splitting for some splitting rate $b \in \mathbb{M}^{d_2 \times d_1}$, if and only if there exists a local isometry $B : L^2\big(G^{(d_1)}\big) \to L^2\big(G^{(d_2)}\big)$ such that $V = \Gamma(B)$. In this case the matrices $b(x)$ are isometric for μ-a.e. $x \in G$ and $B = O_b$.*

Proof: Necessity: If V is a beam splitting, $V = \mathcal{V}_b = \Gamma(O_b)$ and the matrices $b(x)$ are isometric for μ-a.e. $x \in G$ by definition. Also O_b is local by Theorem 3.2.7 and isometric by property (4) of Lemma 3.2.3.

Sufficiency: If B is local then there exists $b \in \mathbb{M}^{d_2 \times d_1}$ such that $B = O_b$ by Theorem 3.2.7. The matrices $b(x)$ are isometric for μ-a.e. $x \in G$ by property (4) of Lemma 3.2.3. Hence $V = \Gamma(B) = \Gamma(O_b) = \mathcal{V}_b$ is a beam splitting by definition. ■

Remark 3.3.4 *Using identification*

$$\mathcal{M}\big(G^{(d_i)}\big) \cong \Gamma\left(L^2(G, \mathbb{C}^{d_i})\right), \tag{3.3.3}$$

an operator $V : \Gamma\left(L^2(G, \mathbb{C}^{d_1})\right) \to \Gamma\left(L^2(G, \mathbb{C}^{d_2})\right)$ is a beam splitting for some splitting rate $b \in \mathbb{M}^{d_2 \times d_1}$, if and only if there exists a local isometry $B : L^2(G, \mathbb{C}^{d_1}) \to L^2(G, \mathbb{C}^{d_2})$ such that $V = \Gamma(B)$. In this case the matrices $b(x)$ are isometric for μ-a.e. $x \in G$ and $B = O_b$.

Theorem 3.3.5 *For $d_1 \leq d_2 \in \mathbb{N}$ and a vacuum-preserving isometry $V : \mathcal{M}\big(G^{(d_1)}\big) \to \mathcal{M}\big(G^{(d_2)}\big)$, the following conditions are equivalent:*

(1) $V = V_{|A} \otimes V_{|A^c}$ for all $A \in \mathfrak{G}$.

(2) $V = V_A \otimes V_{A^c}$ for all $A \in \mathfrak{G}$, where V_A, V_{A^c} are vacuum-preserving isometries.

(3) V is $\mathfrak{G}$-factorisable.

(4) There exists an everywhere-defined map $T : L^2\big(G^{(d_1)}\big) \to L^2\big(G^{(d_2)}\big)$ such that for all $f \in L^2\big(G^{(d_1)}\big)$ and $A \in \mathfrak{G}$

$$V \exp[f] = \exp[T(f)] \qquad \text{and} \qquad T(\chi_A f) = \chi_A T(f).$$

(5) There exists an isometry $T : L^2\big(G^{(d_1)}\big) \to L^2\big(G^{(d_2)}\big)$ such that for all $f \in L^2\big(G^{(d_1)}\big)$ and $A \in \mathfrak{G}$

$$V \exp[f] = \exp[Tf] \qquad \text{and} \qquad T(\chi_A f) = \chi_A T(f).$$

(6) $V = V_b = \Gamma(O_b)$ *is a beam splitting with* d_1 *beams of input,* d_2 *beams of output and splitting rate* $b \in \mathbb{M}^{d_2 \times d_1}$, *where the matrices* $b(x)$ *are isometric for* μ*-a.e.* $x \in G$.

Proof: (1) implies (2): $V_{|A}$ and $V_{|A^c}$ are again isometric and vacuum-preserving.

(2) implies (3): Lemma 2.3.5 and Definition 2.1.3.

(3) implies (4) implies (1): Proposition 2.3.2 and remark 2.3.4.

(4) equivalent (5): Proposition 2.6.3 shows necessity. Sufficiency is clear.

(5) equivalent (6): Theorem 3.3.3. ■

We summarise the definition and characterisation of beam splittings in the following table:

Level	Definition (downwards)	Characterisation (upwards)
local (pointwise)	isometric matrices $b(x)$	isometric matrices $b(x)$
	$\downarrow$	$\uparrow$
global, one particle	isometric operator of matrix multiplication O_b	$T = O_b$ $\Uparrow$ isometry T, preserves regional subspaces
	$\downarrow$	$\uparrow$
particle configurations	beam splitting $\mathcal{V}_b = \Gamma(O_b)$	$\mathcal{V} = \Gamma(T)$ $\Uparrow$ factorisable, vacuum-preserving isometry $\mathcal{V}$

Table 3.1: Overview: Definition/Characterisation of Beam Splitting

3.4 Beam Splittings as Operators of Independent Exchange

In this section beam splittings with $d = 2$ beams of in- and output are reviewed to be a special case of exchange operators dicussed in [25]. To this end we mention two unbounded operators dealed with in detail in [21, 28, 36]. The first one, called compound Malliavin derivative, $\mathcal{D}^c : \mathcal{M}(G) \to \mathcal{M}^{\otimes 2}(G)$ is defined through

$$\mathcal{D}^c\psi(\varphi_1, \varphi_2) := \psi(\varphi_1 + \varphi_2) \qquad \big(\psi \in \mathrm{dom}(\mathcal{D}^c)\,, \varphi_1, \varphi_2 \in M(G)\big). \tag{3.4.1}$$

It is a closed operator with maximal domain given by $\psi \in \mathrm{dom}(\mathcal{D}^c)$ if and only if

$$\int 2^{\varphi(G)} \left|\psi(\varphi)\right|^2 \, dF_\mu(\varphi) < \infty. \tag{3.4.2}$$

Since $2^{\varphi(G)} = \exp[2]\,(\varphi)$ for all $\varphi \in M(G)$, (3.4.2) is equivalent to

$$\mathrm{dom}(\mathcal{D}^c) = \mathrm{dom}\Big(O_{\exp\left[\sqrt{2}\right]}\Big)\,, \tag{3.4.3}$$

which, together with (1.4.5), shows that $\mathrm{dom}(\mathcal{D}^c)$ contains all exponential vectors. On these, by (3.4.1) and definition of the coherent function $\big($see (1.4.1)$\big)$, $\mathcal{D}^c$ acts as

$$\mathcal{D}^c \exp[f] = \exp[f] \otimes \exp[f] \qquad \big(f \in L^2(G)\big). \tag{3.4.4}$$

The compound Skorohod integral, $\mathcal{S}^c : \mathcal{M}^{\otimes 2}(G) \to \mathcal{M}(G)$ is defined through

$$\mathcal{S}^c\psi(\varphi) := \sum_{\varphi_1+\varphi_2=\varphi} \psi(\varphi_1, \varphi_2) \qquad \big(\psi \in \mathrm{dom}(\mathcal{S}^c)\,, \varphi \in M(G)\big), \tag{3.4.5}$$

on the maximal domain given by $\psi \in \mathrm{dom}(\mathcal{S}^c)$ if and only if

$$\int 2^{\varphi_1(G)+\varphi_2(G)} \left|\psi(\varphi_1, \varphi_2)\right|^2 \, dF_\mu^{\otimes 2}(\varphi_1, \varphi_2) < \infty, \tag{3.4.6}$$

which is equivalent to

$$\mathrm{dom}(\mathcal{D}^c) = \mathrm{dom}\Big(O_{\exp\left[\sqrt{2}\right]\otimes\exp\left[\sqrt{2}\right]}\Big)\,. \tag{3.4.7}$$

Hence, $\mathrm{dom}(\mathcal{S}^c)$ contains all tensor products of exponential vectors and, by (3.4.5) and (1.4.6), acts on them according to

$$\mathcal{S}^c \exp[f] \otimes \exp[g] = \exp[f + g] \qquad \big(f, g \in L^2(G)\big). \tag{3.4.8}$$

$\mathcal{D}^c$ and $\mathcal{S}^c$ are mutually adjoint. We have

$$\langle\mathcal{D}^c\psi, \phi\rangle_{\mathcal{M}^{\otimes 2}(G)} = \langle\psi, \mathcal{S}^c\phi\rangle_{\mathcal{M}(G)} \qquad \big(\psi \in \mathrm{dom}(\mathcal{D}^c)\,, \phi \in \mathrm{dom}(\mathcal{S}^c)\big). \tag{3.4.9}$$

Remark 3.4.1 *Both the compound Malliavin derivative $\mathcal{D}^c$ and the compound Skorohod integral $\mathcal{S}^c$ owe their name to the well-known isomorphism between Wiener space and the symmetric Fock space $\Gamma\big(L^2[0,\infty)\big)$ (see [44, Example 7.1]).*

Remark 3.4.2 *The action of $\mathcal{D}^c$ may be interpreted as follows: The quantum configuration $\varphi = \varphi_1 + \varphi_2$ is split into two parts φ_1 and φ_2, where the function value on the split configuration (φ_1, φ_2) is the same as on the original one φ. On the other hand, $\mathcal{S}^c$ merges two quantum configuration φ_1, φ_2 into one $\varphi = \varphi_1 + \varphi_2$.*

Now for some suitable $v \in \mathbb{M}\big([M(G)]^4\big)$ consider the operator U_v on $\mathcal{M}^{\otimes 2}(G)$ acting as

$$U_v := \mathcal{S}^c \otimes \mathcal{S}^c T O_v \mathcal{D}^c \otimes \mathcal{D}^c, \tag{3.4.10}$$

where T is the map that exchanges the second and third argument of a function $v \in \mathbb{M}\big([M(G)]^4\big)$, i.e.

$$Tv(\varphi_1, \varphi_2, \varphi_3, \varphi_4) := v(\varphi_1, \varphi_3, \varphi_2, \varphi_4) \qquad \big(\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in M(G)\big). \tag{3.4.11}$$

In particular, if v is chosen such that

$$O_v = O_{\exp[\alpha]} \otimes O_{\exp[\beta]} \otimes O_{\exp[\alpha']} \otimes O_{\exp[\beta']} \tag{3.4.12}$$

for $\alpha, \alpha', \beta, \beta' \in L^\infty(G)$, then U_v acts on exponential vectors according to

$$\begin{aligned} &U_v \exp[f] \otimes \exp[g] \\ &= \mathcal{S}^c \otimes \mathcal{S}^c T O_v \mathcal{D}^c \otimes \mathcal{D}^c \exp[f] \otimes \exp[g] && \text{by (3.4.10)} \\ &= \mathcal{S}^c \otimes \mathcal{S}^c T O_v \exp[f] \otimes \exp[f] \otimes \exp[g] \otimes \exp[g] && \text{by (3.4.4)} \\ &= \mathcal{S}^c \otimes \mathcal{S}^c T \exp[\alpha f] \otimes \exp[\beta f] \otimes \exp[\alpha' g] \otimes \exp[\beta' g] && \text{by (3.4.12)} \\ &= \mathcal{S}^c \otimes \mathcal{S}^c \exp[\alpha f] \otimes \exp[\alpha' g] \otimes \exp[\beta f] \otimes \exp[\beta' g] && \text{by (3.4.11)} \\ &= \exp[\alpha f + \alpha' g] \otimes \exp[\beta f + \beta' g] && \text{by (3.4.8),} \end{aligned} \tag{3.4.13}$$

i.e. considering U_v as an operator on $\mathcal{M}\big(G^{(2)}\big) \cong \mathcal{M}^{\otimes 2}(G)$ and comparing (3.4.13) with (3.2.1) we find that it extends to

$$U_v = \Gamma(O_b) \qquad \text{for} \qquad b(x) = \begin{pmatrix} \alpha(x) & \alpha'(x) \\ \beta(x) & \beta'(x) \end{pmatrix} \qquad (x \in G). \tag{3.4.14}$$

If the matrices $b(x)$ are unitary for μ-almost every $x \in G$ then U_v is a beam splitting with $d = 2$ beams of in- and output.

Because of the tensor product structure of O_v in (3.4.12), U_v is called operator of independent exchange in [25]. For more general v the operators U_v have the property

$$O_{\mathcal{D}^c\psi} U_v = U_v O_{\mathcal{D}^c\psi} \qquad \Big(\psi \in L^\infty\big(M(G)\big)\Big). \tag{3.4.15}$$

$O_{\mathcal{D}^c\psi}$ represents the measurement of ψ on the compound system, ignoring whether particles belong to the first or the second subsystem. Therefore, (3.4.15) means that U_v does not affect the measurement of ψ. It can be seen as an exchange of particles from the two subsystems. Hence, operators U_v are called exchange operators. In fact, they are the only operators that fulfill (3.4.15). We have

Proposition 3.4.3 *(see [25, Theorem 9]) For finite μ, a bounded operator U on $\mathcal{M}^{\otimes 2}(G)$ commutes with $O_{\mathcal{D}^c\psi}$ for all $\psi \in L^\infty\big(M(G)\big)$ if and only if $U = U_v$ for some $v \in V$ and a suitable class of functions V.*

3.5 Beam Splittings in Brain Modelling

> *The light of the body is the eye: therefore when thine eye is single, thy whole body also is full of light; (Luke 11:34)*

Studying $\mathfrak{G}$-factorisable operators on multliple Fock space was motivated by a quantum model of recognition described in [29, 11, 12, 13, 14, 10, 33, 43, 15], which was developed in close cooperation with neuroscientists. We will briefly recapitulate their requirements on such a model in the form of postulates from [15]:

(P1) *The brain acts discrete in time.*

(P2) *Signals are represented by populations of excited neurons.*

(P3) *Signals can be decomposed into parts in compliance with the fact that there are different regions of the brain responsible for different tasks.*

(P4) *The brain acts parallel corresponding to the different regions. Moreover this action is governed by the same principles independent of the region: "a unified algorithm" (see postulate P4 in [33]).*

(P5) *Signals stored in the brain are superpositions of finitely many elementary signals.*

(P6) *The brain permanently creates complex signals representing an "expected view of the world".*

(P7) *Recognition of a signal produced by our senses is a random event which can occur as a consequence of the interaction of that signal and a signal created by the brain.*

(P8) *Recognition causes a loss of excited neurons in some region of the brain.*

(P9) *Recognition changes the state of the signal coming from our senses. One will be aware of that changed signal.*

(P10) *Changes in some region of the brain have immediate consequences in other regions.*

In this model, signals are represented by (pure) states on the symmetric Fock space $\mathcal{M}(G), G$ being the physical space where recognition takes place, i.e. the brain. For example, take G to be a bounded subset of $\mathbb{R}^3$ and μ Lebesque measure. Remembering that $\mathcal{M}(G)$ is an L^2-space over point configurations, which can be interpreted as the positions where the quantum amounts of energy of the excited neurons are located, this is in accordance with P2.

Interaction of two signals, one arising from the senses and the other generated from the memory (representing an “expected view of the world”, see P6 and P7), is therefore modelled by a unitary operator $\mathcal{V}$ on

$$\mathcal{M}(G) \otimes \mathcal{M}(G) \cong \mathcal{M}\big(G^{(2)}\big) \cong \Gamma\big(L^2(G, \mathbb{C}^2)\big) \cong \Gamma\big(L^2(G)\big) \otimes \Gamma\big(L^2(G)\big).$$

Neuroscience suggests (see P3 and P4) that the operator $\mathcal{V}$ factorises according to

$$\mathcal{V} = \bigotimes_{k=1}^{n} \mathcal{V}_{G_k} \qquad \big(G_1, \ldots, G_n \in \Sigma(G)\big), \tag{3.5.1}$$

which corresponds to a parallel processing of partial signals belonging to different regions of the brain. Thereby, requiring (3.5.1) to hold for any finite measurable decomposition $G_1, \ldots, G_n$ of G, and not just a fixed one, accounts for the fact that decomposition of G into different regions not only depends on the individual but also may change over time. Furthermore, interaction in the different regions is governed by the same principles, ”a unified algorithm” (P4). Mathematically, this leads to $\mathcal{V}_{G_k} = \mathcal{V}_{|G_k}$.

We also require $\mathcal{V}$ to preserve the vacuum, i.e.

$$\mathcal{V} \exp[0] \otimes \exp[0] = \exp[0] \otimes \exp[0]\,, \tag{3.5.2}$$

which has a reasonable interpretation: If both memory and senses produce the empty signal, the organism is dead. Hence there is no more change in this state.

Theorem 3.3.5 shows, that beam splittings are the only candidates for this kind of interaction operator. By reasons beyond the scope of this paper (for example symmetry and homogenity conditions), the operator used in the model is the so-called symmetric beam splitting given by $b_{1,1} \equiv b_{1,2} \equiv b_{2,1} \equiv -b_{2,2} \equiv \frac{\sqrt{2}}{2}$. On exponential vectors it acts according to

$$\mathcal{V} \exp[f] \otimes \exp[g] = \exp\Big[\tfrac{\sqrt{2}}{2}(f+g)\Big] \otimes \exp\Big[\tfrac{\sqrt{2}}{2}(f-g)\Big] \qquad \big(f, g \in L^2(G)\big). \tag{3.5.3}$$

We observe three important properties of this particular symmetric beam splitting $\mathcal{V}$:

1. $\mathcal{V}$ is both unitary and self-adjoint. This implies $\mathcal{V}\mathcal{V} = \mathbb{1}_{\mathcal{M}(G)\otimes\mathcal{M}(G)}$. Applying $\mathcal{V}$ a second time restores the initial signals.

2. If both initial signals coincide, after interaction the second component will be equal to the vacuum:

$$\mathcal{V} \exp[f] \otimes \exp[f] = \exp\Big[\sqrt{2}f\Big] \otimes \exp[0] \qquad \big(f \in L^2(G)\big). \tag{3.5.4}$$

3. If the second signal is the vacuum signal, after processing both signals will coincide:

$$\mathcal{V}\exp[f]\otimes\exp[0]=\exp\left[\tfrac{\sqrt{2}}{2}f\right]\otimes\exp\left[\tfrac{\sqrt{2}}{2}f\right]\qquad\big(f\in L^2(G)\big). \tag{3.5.5}$$

This gives rise to the following description of the process of recognition:

The event of recognition occurs, if both initial signals $\exp[f]$ and $\exp[g]$ coincide. This event is represented by the projection onto the vacuum in the second component after interaction of the two signals according to $\mathcal{V}$. The probability of recognition is therefore equal to $\tau(\mathrm{Proj}_{\exp[0]})$, where τ is the pure state given by the wave function $\widehat{\Phi}=\frac{\Phi}{\|\Phi\|}$ with

$$\Phi=\exp\left[\tfrac{\sqrt{2}}{2}(f-g)\right],\qquad\text{i.e.}\qquad\widehat{\Phi}=e^{-\frac{1}{4}\|f-g\|^2}\exp\left[\tfrac{\sqrt{2}}{2}(f-g)\right].$$

Hence, recognition occurs with probability

$$\tau(\mathrm{Proj}_{\exp[0]})=\left\langle\widehat{\Phi},\mathrm{Proj}_{\exp[0]}\widehat{\Phi}\right\rangle=e^{-\frac{1}{2}\|f-g\|^2}. \tag{3.5.6}$$

After recognition the second signal will be in the vacuum state (self-collapse, causing a loss of excited neurons, see P8). Another application of $\mathcal{V}$ will then lead to

$$\mathcal{V}\exp\left[\tfrac{\sqrt{2}}{2}(f+g)\right]\otimes\exp[0]=\exp\left[\tfrac{1}{2}(f+g)\right]\otimes\exp\left[\tfrac{1}{2}(f+g)\right]. \tag{3.5.7}$$

Both signals coincide according to a combination of the initial two. This is the perceived signal (P9). Now repeating the procedure (consecutively applying $\mathcal{V}$, then the projection onto the vacuum in the second component and again $\mathcal{V}$) will cause no more change of this signal, because recognition will now always occur with probability one (no more collapse).

But what if recognition did not occur, which happens with probability $1-e^{-\frac{1}{2}\|f-g\|^2}$? Then applying $\mathcal{V}$ a second time will restore the initial signals and the procedure can start all over.

Now if we want to incorporate other aspects of the process of recognition, for example the influence of short-term memory or intentional seeking for something, we must be able to describe an interaction of more than just two signals. Hopefully, beam splittings with an arbitrary number of beams of in- and output will help solve this problem.

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List of Symbols

General

Numbers

$\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	sets of natural, non-negative integer, integer, rational, real and complex numbers
$\operatorname{Re} z, \operatorname{Im} z, \overline{z}$	real, imaginary part and complex conjugate of $z \in \mathbb{C}$

Operations

$\oplus$	orthogonal sum of spaces or vectors
$\otimes$	(tensor) product of spaces, vectors, operators or measures
$(\cdot)^{\otimes n}$	n-fold (tensor) power of a space, vector, operator or measure
$\times$	cartesian product of sets

Sets

Y^d	cartesian power of a set Y
Y^c	complement of a measurable set Y
χ_Y	indicator function of a set Y
$\chi_Y^c = 1 - \chi_Y$	indicator function of Y^c

Spaces

$L^\infty(\cdot)$	space of essentially bounded, measurable maps
$\mathbb{M}(\Omega, \Omega')$	set of measurable maps from Ω to Ω'
$\mathbb{M}(\Omega)$	set of measurable maps from Ω to $\mathbb{C}$

Hilbert Spaces

H	separable Hilbert space
$\langle \cdot, \cdot \rangle_H$	scalar product in H, linear in second argument
$\|\cdot\|_H$	norm induced by scalar product in H
$S^\perp$	orthogonal complement of $S \subset H$
$\mathrm{Lin}(S)$	linear space generated by $S \subset H$
$\cong$	isomorphic equivalence of Hilbert spaces, vectors or operators
$\mathcal{B}(H)$	set of bounded linear operators on H
B^*	adjoint of an (possibly unbounded) operator B
$\mathrm{dom}(B)$	maximal domain of unbounded operator B
$\mathbb{1}_H$	identity operator on H
$\mathrm{Proj}_{H'}$	orthogonal projection onto a subspace $H' \subset H$
Proj_f	orthogonal projection onto $H' = \mathrm{Lin}(\{f\})$
$\widehat{f} := f / \|f\|$	normalised $0 \neq f \in H$, a wave function
τ	state on $\mathcal{B}(H)$
τ_ϱ	normal state with density matrix ϱ
τ_f	pure state given by wave function f
Q_τ	position distribution of normal state τ
Q_τ^0	avoidance function of position distribution of normal state τ on Fock space
$L^2(\cdot)$	Hilbert space of square-integrable maps
O_g	operator of multiplication with g
$O_A := O_{\chi_A}$	operator of multiplication with indicator function χ_A, represents position measurement
$L^2(\cdot, \mathbb{C}^d)$	Hilbert space of $\mathbb{C}^d$-valued, square-integrable maps
O_b	operator of matrix-multiplication with matrix-valued b

Location Space

G	separable and metric space, space, where point configurations are located
$\mathfrak{G}$	σ-algebra of Borel sets over G
μ	finite, non-atomic measure on $(G, \mathfrak{G})$
A	measurable subset of G, i.e. $A \in \mathfrak{G}$
$A \cap \mathfrak{G}$	spur σ-algebra of $\mathfrak{G}$ with respect to $A \in \mathfrak{G}$
$\mu_{\vert A}$	restriction of μ to $A \cap \mathfrak{G}$, $A \in \mathfrak{G}$
$\Sigma(A)$	set of finite measurable decompositions of $A \in \mathfrak{G}$
$\sigma = (G_1, \ldots, G_n)$	a finite measurable decomposition of G, i.e. $\sigma \in \Sigma(G)$
$A^{(d)} = A \times \{1, \ldots, d\}$	compound of d copies of A
$\mu_{\vert A}^{(d)} = \mu_{\vert A} \otimes \sum_{i=1}^{d} \delta_i$	compound of d copies of $\mu_{\vert A}$
$\chi_A = \chi_{A^{(d)}}$	short-hand, meaning $\chi_A f = (\chi_A f_1, \ldots, \chi_A f_d)$
$f_{\vert A}$	restriction of $f \in \mathbb{M}(G)$ to $A \in \mathfrak{G}$
$f_{\vert A} = f_{\vert A^{(d)}}$	restriction of $f \in \mathbb{M}\big(G^{(d)}\big)$ to $A^{(d)}$
$\mathbb{M}$	set of measurable maps from G to $\mathbb{C}$
$\mathbb{M}^m$	set of measurable maps from G to $\mathbb{C}^m$
$\mathbb{M}^{m \times n}$	set of measurable maps from G to $\mathbb{C}^{m \times n}$

Configuration Space (for $d = 1$ drop the superscript)

Configurations

$\mathcal{O}$	empty configuration
δ_x	one-point configuration in x
$\delta_{(x,i)}$	one-point configuration, x belongs to the i-the part
$\varphi = (\varphi_1, \ldots, \varphi_d)$	locally finite, d-tuple point configuration
$\varphi_{\vert A} = \varphi\big(\cdot \cap A^{(d)}\big)$	restriction of φ to points in A
$\hat{\varphi} \leq \varphi$	$\hat{\varphi}$ is subconfiguration of φ
$x \in \varphi$	x is a point of the single configuration $\varphi \in M(G)$
$(x, i) \in \varphi$	x is a point of the multiple configuration $\varphi \in M\big(G^{(d)}\big)$, in particular, $x \in \varphi_i$

Sets of Configurations

$M\big(A^{(d)}\big)$	(set of) locally finite, d-tuple configurations with points in A
$M_n\big(A^{(d)}\big)$	d-tuple configurations in A, with exactly n points
$M_{\leq n}\big(A^{(d)}\big)$	d-tuple configurations in A, having at most n points
$M_{<\infty}\big(A^{(d)}\big)$	finite, d-tuple configurations with points in A
$M^s\big(A^{(d)}\big)$	simple, d-tuple configurations with points in A
$M^s_{<\infty}\big(A^{(d)}\big)$	simple, finite, d-tuple configurations with points in A
$\mathfrak{M}\big(A^{(d)}\big)$	σ-algebra over $M\big(A^{(d)}\big)$

Fock Space (for $d = 1$ drop the superscript)

Measures, Spaces, Functions

$F_{\mu^{(d)}}$	multiple Fock space measure on $M\big(G^{(d)}\big)$
$F_{\mu^{(d)}_{\vert A}}$	multiple Fock space measure on $M\big(A^{(d)}\big)$
$F^{\otimes d}_{\mu_{\vert A}}$	d-fold product of Fock space measure on $[M(A)]^d$
$\Gamma(H)$	symmetric Fock space over H
$\mathcal{M}(A^{(d)}) = L^2\big(M(A^{(d)})\big)$	multiple symmetric Fock space over $L^2\big(A^{(d)}\big)$
$\Psi = (\Psi_n), \Phi = (\Phi_n)$	elements of $\Gamma(H)$ or $\mathcal{M}(G)$
$\psi_{\vert A} = \psi_{\vert M(A^{(d)})}$	restriction of $\psi \in \mathcal{M}(G^{(d)})$ to configurations in A

Exponential Vectors

$\mathbb{C}\exp[H]$	set of complex multiples of exponential vectors
$\exp[f]$	exponential vector generated by f
$\exp[f]\,(\varphi)$	coherent function evaluated at φ
$\exp[H]$	exponential vectors generated from all $f \in H$
$\mathcal{E}(H) = \mathrm{Lin}(\exp[H])$	exponential domain, linear space generated by the exponential vectors
$h_{G_1,\dots,G_n}(z, f, g)$	function used to approximate $\exp[zf + zg]$, see (2.5.15)
$h_A(z, f, g)$	function used to approximate $\exp[zf + zg]$, see (2.5.16)

Operators

$B_{\vert A} = B_{\vert \mathcal{M}(A^{(d)})}$	restriction of operator B on $\mathcal{M}\big(G^{(d)}\big)$ to $\mathcal{M}\big(A^{(d)}\big)$
$\mathcal{D}^c$	compound Malliavin derivative
$\mathcal{S}^c$	compound Skorohod integral
$\Gamma(T)$	operator of second quantisation of T
$\mathcal{V}_b = \Gamma(O_b)$	beam splitting with splitting rate b
$\mathcal{W}\,(f)$	Weyl operator associated with f